ABELIAN STRICT APPROXIMATION IN AW*-ALGEBRAS AND WEYL-VON NEUMANN TYPE THEOREMS

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Dedicated to Professor E. Effros on his 70^{th} *birthday*

Abstract. In this paper, for a C^*-algebra A with $M = M(A)$ an AW^*-algebra, or equivalently, for an essential, norm-closed, two-sided ideal A of an AW^*-algebra M, we investigate the strict approximability of the elements of M from commutative C^*-subalgebras of A. In the relevant case of the norm-closed linear span A of all finite projections in a semi-finite AW^*-algebra M we shall give a complete description of the strict closure in M of any maximal abelian self-adjoint subalgebra (masa) of A. We shall see that the situation is completely different for discrete respectively continuous M:

In the discrete case, for any masa C of A, the strict closure of C is equal to the relative commutant $C' \cap M$, while in the continuous case, under certain conditions concerning the center valued quasitrace of the finite reduced algebras of M (satisfied by all von Neumann algebras), C is already strictly closed. Thus in the continuous case no elements of M which are not already belonging to A can be strictly approximated from commutative C^*-subalgebras of A.

In spite of this pathology of the strict topology in the case of the norm-closed linear span of all finite projections of a continuous semi-finite AW^*-algebra, we shall prove that in general situations including also this case, any normal $y \in M$ is equal modulo A to some $x \in M$ which belongs to an order theoretical closure of an appropriate commutative C^*-subalgebra of A. In other words, if we replace the strict topology with order theoretical approximation, Weyl-von Neumann-Berg-Sikonia type theorems will hold in substantially greater generality.

Introduction

Let A be a C^*-algebra. The *multiplier algebra* of A is the C^*-subalgebra

$$\{x \in A^{**};\ xa,\ ax \in A \text{ for all } a \in A\}$$

of the second dual A^{**} (see [Ped 2], Section 3.12 or [WO], Chapter 2). A natural locally convex vector space topology on $M(A)$, called the *strict topology* β, is defined by the seminorms

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$$x \mapsto \|xa\| \text{ and } x \mapsto \|ax\|, \qquad a \in A.$$

It is complete and compatible with the duality between $M(A)$ and A^*. Hence the strict topology is weaker than the norm-topology on $M(A)$, but stronger than the restriction to $M(A)$ of the weak * topology of A^{**}.

We notice that for A the C^*-algebra $K(H)$ of all compact linear operators on a complex Hilbert space H, $M(A)$ can be identified with the C^*-algebra $B(H)$ of all bounded linear operators on H and on every bounded subset of $B(H)$ the strict topology coincides with the s^*-topology.

More generally, if M is an AW^*-algebra (see [Kap 1] or [Be], §4 or [S-Z], §9) and A is an essential, norm-closed, two-sided ideal of M, then, by a theorem of B. E. Johnson, M can be identified with $M(A)$ (see [J] or [Ped 3]). Thus the pairs $\big(A, M(A)\big)$, where A is a C^*-algebra such that $M(A)$ is an AW^*-algebra, are exactly the pairs $\big(A, M\big)$, where M is an AW^*-algebra and A is an essential, norm-closed, two-sided ideal of M.

A relevant case of essential, norm-closed, two-sided ideal of an AW^*-algebra is the norm-closed linear subspace A generated by all finite projections of a semi-finite AW^*-algebra M. Then there are central projections p_1, p_2, p_3 of M with $p_1 + p_2 + p_3 = 1_M$ such that Mp_1 is finite, Mp_2 is properly infinite and discrete, while Mp_3 is properly infinite and continuous (see [Be], §15, Theorem 1). Since $Ap_1 = Mp_1$, the non-trivial cases are Ap_2 and Ap_3, with $M(Ap_2) = Mp_2$ properly infinite and discrete and $M(Ap_3) = Mp_3$ properly infinite and continuous.

In the previous paper [D-Z] we investigated the strict approximability of a normal element x of $M(A)$ from a commutative C^*-subalgebra of A. More precisely, we say that x belongs to the *abelian strict closure* of A if there exists a commutative C^*-subalgebra C_x of A such that $x \in \overline{C_x}^{\beta}$. Abelian strict approximability is closely related to the classical Weyl-von Neumann-Berg-Sikonia (WNBS) Theorem, which claims that in the case of $A = K(H)$, H a separable complex Hilbert space, every normal element of $M(A) = B(H)$ is of the form $a+x$ with $a \in A$ and x in the abelian strict closure of A. For a general σ-unital C^*-algebra A, that is a C^*-algebra having a countable approximate unit, we proved a partial extension in [D-Z], Theorem 1, which implies that all elements $y \in M(A)$ are of the form $a + x_1 + x_2$, where $a \in A$ and $x_1 \in B_1$, $x_2 \in B_2$, where B_1, B_2 are separable C^*-subalgebras of $M(A)$ such that every normal element of B_j, $j = 1, 2$, belongs to the abelian strict closure of A. Moreover, if y is self-adjoint then x_1, x_2 can be chosen self-adjoint, so in this situation x_1, x_2 themselves belong to the abelian strict closure of A.

We notice that if the multiplier algebra of a σ-unital C^*-algebra A is of real rank zero (see [Br-Ped]), then, according to [M] and [Zh], the WNBS Theorem holds in the same formulation as in the classical case.

In this paper we discuss abelian strict approximability for a C^*-algebra A which is the norm-closed linear subspace generated by all finite projections of some semi-finite AW^*-algebra M. Since the abelian strict closure of A is the union of all $\overline{C}^{\beta}$ with C a maximal abelian self-adjoint subalgebra (*masa*) of A, we are interested in describing $\overline{C}^{\beta}$ for any masa C of A. We shall see that the situation is completely different for discrete respectively continuous $M(A) = M$:

In the discrete case $\overline{C}^{\beta}$ is equal to the relative commutant $C' \cap M(A)$ (Theorem 1), while in the continuous case, under a certain condition on the centre valued quasitrace of the reduced AW^*-subalgebra of $M(A)$ by a finite projection of central

support $1_{M(A)}$ (always satisfied if $M(A)$ is a von Neumann algebra), C is already strictly closed (Theorem 3).

Consequently, if $M(A)$ is a properly infinite, continuous AW^*-algebra satisfying the above mentioned condition, then the unit of $M(A)$ does not belong to the abelian strict closure of A, that is it does not exist an approximate unit for A contained in a commutative $*$-subalgebra of A. In particular, in this case A is not σ-unital. We notice that it was already shown in [Ak-Ped], Proposition 4.5, that the norm-closed linear span of all finite projections of a type II_∞ factor is a non-σ-unital C^*-algebra. Nevertheless, also in this case WNBS type theorems can be proved. Indeed, if A is the norm-closed linear subspace generated by all finite projections of some countably decomposable semi-finite W^*-factor M, then, according to [Z], Theorem 3.1, every normal $y \in M(A) = M$ is of the form $a+x$ with $a \in A$ and x in the s^*-closure in M of some masa C of A. Since the s^*-closure of a commutative $*$-subalgebra of a W^*-algebra is equal to its monotone order closure (cf. [Kad 1] and [Ped 1]), it is natural to expect that for extensions of the WNBS Theorem to non-σ-unital C^*-algebras the strict closure should be replaced by an order theoretical closure. Along this line we prove several WNBS type theorems in a general setting which includes the case of the norm-closed linear span of all finite projections of a countably decomposable semi-finite AW^*-algebra.

More precisely, we prove that if $\mathcal{J}$ is a norm-closed two-sided ideal of a (unital) Rickart C^*-algebra M, which has a countable "order theoretical approximate unit", then any normal $y \in M$ is of the form $y = a + x$, where $a \in A$ is of arbitrarily small norm and x belongs to the order theoretical closure of some masa of $\mathcal{J}$ (Theorem 4 and the subsequent remark). Moreover, the above x can be chosen as a particular infinite linear combination of a sequence of mutually orthogonal projections from $\mathcal{J}$ (Theorems 5 and 6).

Since only little of the specific properties of Rickart C^*-algebras is used, we are left with the question, to which extent the above mentioned WNBS type theorems hold if M is assumed to be only a C^*-algebra of real rank zero.

1 Abelian Strict Closure in Discrete AW*-algebras

First we prove a general result concerning a masa C of a C^*-algebra A, whose multiplier algebra is an AW^*-algebra, that is, according to the theorem of B. E. Johnson quoted in Introduction (see [J] or [Ped 3]), a masa C of an essential, norm-closed, two-sided ideal A of some AW^*-algebra. We notice that a part of this result holds for a masa of an essential, norm-closed, two-sided ideal of any Rickart C^*-algebra. We shall restrict us to unital Rickart C^*-algebras, because adjoining a unit to a non-unital Rickart C^*-algebra M, we obtain a unital Rickart C^*-algebra $\widetilde{M}$ (see [Be], §5, Theorem 1 or [S-Z], 9.11.(1)) and it is easy to see that every essential, norm-closed, two-sided ideal of M is an essential, norm-closed, two-sided ideal also of $\widetilde{M}$.

Any essential two-sided ideal $\mathcal{J}$ of a C^*-algebra M induces a strict topology $\beta_{\mathcal{J}}$ on M, defined by the seminorms

$$M \ni x \mapsto \|xa\| \text{ and } x \mapsto \|ax\|, \qquad a \in \mathcal{J}.$$

With this definition, the usual strict topology on the multiplier algebra of a C^*-algebra A is β_A.

For the basic facts concerning Rickart C^*-algebras and AW^*-algebras see [Be], §§ 3, 4 and 5, or [S-Z], §9.

Lemma 1. *Let M be a unital C^*-algebra, $\mathcal{J}$ an essential, norm-closed, two-sided ideal of M, and C a masa of $\mathcal{J}$. By the strict topology on M we shall understand $\beta_{\mathcal{J}}$, which of course is the usual strict topology when M is an AW^*-algebra and so can be identified with the multiplier algebra $M(\mathcal{J})$. Then*

(i) *every $x \geq 0$ in the strict closure of C in M belongs to the strict closure of $\{b \in C\,; 0 \leq b \leq x\}$ in M.*

Let us next assume that M is a Rickart C^*-algebra. Then

(ii) *for every $0 \leq b \in C$ and every $\delta > 0$ there is a projection $f_\delta \in C$ such that*
$$b f_\delta \geq \delta f_\delta, \quad b(1_M - f_\delta) \leq \delta(1_M - f_\delta)\,,$$
so C is the norm-closed linear span of its projections;

(iii) *any projection e in the strict closure of C in M belongs to the strict closure of $\{f \in C\,; f \leq e$ projection $\}$ in M;*

(iv) *any projection e in the relative commutant $C' \cap M$ is the least upper bound of $\{f \in C\,; f \leq e$ projection $\}$ in the projection lattice of M, in particular $C' \cap M$ is a masa of M.*

Finally, assuming M to be an AW^*-algebra,

(v) *the relative commutant $C' \cap M$ is the AW^*-subalgebra of M generated by C, so $C' \cap M$ can be identified with $M(C)$;*

(vi) *the strict closure of C in M coincides with $C' \cap M$ if and only if C contains a two-sided approximate unit for $\mathcal{J}$, in which case the strict topology of $M(C) = C' \cap M$ is the restriction of the strict topology of $M(\mathcal{J}) = M$.*

Proof. The strict closure $\overline{C}^{\beta_{\mathcal{J}}}$ of C being an abelian C^*-subalgebra of $M(A)$, we have for every $b \in C$
$$(x-b)^*(x-b) \geq (x - \text{Re } b)^2 \geq \big(x - (\text{Re } b)_+\big)^2 \geq (x - b_o)^2\,,$$
where
$$b_o = \tfrac{1}{2}\Big(x + (\text{Re } b)_+ - |x - (\text{Re } b)_+|\Big)$$
denotes the greatest lower bound of x and $(\text{Re } b)_+$ in the Hermitian part of $\overline{C}^{\beta_{\mathcal{J}}}$. Since
$$0 \leq b_o \leq (\text{Re } b)_+ \in C \subset \mathcal{J}\,,$$
by [Ped 2], Prop. 1.4.5 we have $b_o \in \mathcal{J}$, so
$$b_o \in C' \cap \mathcal{J} = C\,.$$
Thus, for every $a \in \mathcal{J}$ and $b \in C$ we have $\|(x-b)a\| \geq \|(x-b_o)a\|$ for some $0 \leq b_o \leq x$ in C and (i) follows.

For (ii) put
$$f_\delta = \text{ support of } (b - \delta 1_{A^{**}})_+ \text{ in } M\,.$$
Then f_δ commutes with every element of C and
$$b f_\delta \geq \delta f_\delta, \quad b(1_M - f_\delta) \leq \delta(1_M - f_\delta)\,.$$
In particular, $f_\delta \leq \frac{1}{\delta} b \in A$ and [Ped 2], Prop.1.4.5 yields $f_\delta \in \mathcal{J}$. Consequently $f_\delta \in C' \cap A = C$.

For (iii) let $0 \neq a \in \mathcal{J}$ and $\varepsilon > 0$ be arbitrary. According to (i) there exists $0 \leq b \leq e$ in C such that

$$\|(e-b)a\| < \frac{\varepsilon}{2}\,.$$

Further, by (ii) there is a projection $f \in C$ with

$$bf \geq \frac{\varepsilon}{2\|a\|}f, \qquad b(1_{A^{**}} - f) \leq \frac{\varepsilon}{2\|a\|} \cdot (1_{A^{**}} - f)\,.$$

Then $f \leq e$ and $e - f \leq (e - bf)^2$, so

$$\begin{aligned}\|(e-f)a\| = \|a^*(e-f)a\|^{1/2} \leq \\ &\leq \|a^*(e-bf)^2a\|^{1/2} = \|(e-bf)e\| \leq \\ &\leq \|(e-b)e\| + \|b(1_{A^{**}}-f)e\| < \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2\|a\|}\|a\| = \varepsilon\end{aligned}$$

For (iv) we have to show that if a projection $g \in M$ majorizes all projections $C \ni f \leq e$, then $g \geq e$, that is e is equal to the greatest lower bound $e \wedge g$ of e and g in the projection lattice of M. Let us assume that

$$e_o = e - e \wedge g \neq 0\,.$$

Since $\mathcal{J}$ is essential ideal in M, there exists $a \in \mathcal{J}$ with $ae_o \neq 0$. Choosing some $0 < \delta < \|e_o a^* a e_o\|$ and putting

$$e_1 = \text{support of } (e_o a^* a e_o - \delta 1_M)_+ \text{ in } M\,,$$

we have

$$0 \neq e_1 \leq \tfrac{1}{\delta} e_o a^* a e_o \in \mathcal{J}\,.$$

Clearly, $e_1 \leq e_o$ and [Ped 2], Prop. 1.4.5 yields also $e_1 \in \mathcal{J}$. Furthermore, for every projection $f \in C$ we get successively

$$\begin{aligned}&fe \in C' \cap \mathcal{J} = C \text{ and } fe \leq e,\\ &fe \leq e \wedge g, \text{ hence } fe_o = (fe)e_o = 0,\\ &fe_1 = (fe_o)e_1 = 0\,.\end{aligned}$$

Taking into account (ii), it follows that

$$be_1 = 0 \text{ for all } b \in C\,,$$

in particular

$$e_1 \in C' \cap \mathcal{J} = C\,.$$

But then $e_1 \leq e_o \leq e$ implies $e_1 \leq e \wedge g$, which contradicts $0 \neq e_1 \leq e_o = e - e \wedge g$.

In particular, $C' \cap M$ is commutative. For the proof we notice that, since $C' \cap M$ is a Rickart C^*-subalgebra of M (see [Be], §5, Proposition 5 or [S-Z], 9.12.(1)), it is the norm-closed linear span of its projections (see e.g. [S-Z], 9.4) and therefore it is enough to show that any two projections $e_1, e_2 \in C' \cap M$ commute. But the $*$-automorphism $M \ni x \longmapsto (2\,e_2 - 1_M)x(2\,e_2 - 1_M) \in M$ leaves fixed C, hence also the least upper bound of any projection family in C in the projection lattice of M. Therefore it leaves fixed e_1, that is $e_1 e_2 = e_2 e_1$.

Moreover, $C' \cap M$ is a masa of M. Indeed, if $C_o \supset C' \cap M$ is a commutative subalgebra of M, then $C_o \supset C$ and thus we have also $C_o \subset {C_o}' \cap M \subset C' \cap M$.

For (v) we first notice that $C' \cap M$ is an AW^*-subalgebra of M containing C (see [Be], §4, Proposition 8 or [S-Z], 9.24.(1)). Now let N be any AW^*-subalgebra of M containing C. By (iv) N contains all projections from $C' \cap M$, hence $N \supset C' \cap M$. Consequently $C' \cap M$ is the AW^*-subalgebra of M generated by C.

Further, C is a two-sided ideal of $C' \cap M$:

$$b \in C \text{ and } y \in C' \cap M \implies by \in C' \cap \mathcal{J} = C .$$

Moreover, it is essential, because a projection $e \in C' \cap M$ with $Ce = \{0\}$ belongs to the AW^*-subalgebra of $C' \cap M$ generated by C only if $e = 0$. Hence we can identify $C' \cap M$ with $M(C)$ (see [J] or [Ped 3]).

Finally we prove (vi). If the strict closure of C in M is $C' \cap M \ni 1_M$, then there exists a net $(u_\iota)_\iota$ in C with $u_\iota \to 1_M$ strictly in M, that is

$$\|a - u_\iota a\| \to 0 \text{ and } \|a - u_\iota a\| \to 0 \text{ for all } a \in \mathcal{J} .$$

Conversely, let us assume that C contains a two-sided approximate unit $(u_\iota)_\iota$ for $\mathcal{J}$. Then the strict topology β_C of $M(C) = C' \cap M$ agrees with the strict topology $\beta_{\mathcal{J}}$ of $M(\mathcal{J}) = M$ on every norm bounded subset of $C' \cap M$. Indeed, if $(y_\lambda)_\lambda$ is a norm bounded net in $C' \cap M$, convergent to 0 with respect to β_C, and $0 \neq a \in \mathcal{J}$, $\varepsilon > 0$ are arbitrary, then there exists ι_o such that

$$\|y_\lambda\| \cdot \|a - u_{\iota_o} a\| < \frac{\varepsilon}{2} \text{ for all } \lambda ,$$

and then there exists some λ_o with

$$\|y_\lambda u_{\iota_o}\| < \frac{\varepsilon}{2\|a\|} \text{ for every } \lambda \geq \lambda_o .$$

It follows for every $\lambda \geq \lambda_o$:

$$\|y_\lambda a\| \leq \|y_\lambda (a - u_{\iota_o} a)\| + \|y_\lambda u_{\iota_o} a\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2\|a\|} \|a\| = \varepsilon .$$

But β_C is the finest locally convex vector space topology on $C' \cap M$ that agrees with β_C on every norm bounded subset of $C' \cap M$ (see [T], Cor. 2.7). Thus the restriction of $\beta_{\mathcal{J}}$ to $C' \cap M$, which is plainly finer than β_C, is actually equal to β_C. In particular, the β_C-density of C in $M(C)$ implies the $\beta_{\mathcal{J}}$-density of C in $C' \cap M$. □

It is well known that every commutative AW^*-algebra Z is monotone complete (see e.g. [S-Z], 9.26, Proposition 1). If M is an arbitrary AW^*-algebra, we call

$$\Phi : \big\{ e \in M ; e \text{ projection } \big\} \to Z^+$$

completely additive whenever, for every family $(e_\iota)_\iota$ of mutually orthogonal projections in M, we have

$$\Phi\big(\bigvee_\iota e_\iota \big) = \sum_\iota \Phi(e_\iota) ,$$

where the sum stands for the least upper bound in Z^+ of all finite sums of $\Phi(e_\iota)$'s.

Now we describe the strict closure of a masa of the norm-closed two-sided ideal generated by the finite projections of a discrete semi-finite AW^*-algebra :

Theorem 1 (on the abelian strict closure in discrete AW^*-algebras)**.** *Let M be a discrete AW^*-algebra, A the norm-closed linear span of all finite projections of M, and C a masa of A. Then the strict closure of C in $M(A) = M$ is equal to $C' \cap M$.*

Proof. According to Lemma 1 (vi), we have to show that C contains a two-sided approximate unit for A. Without loss of generality we may assume that $A \neq \{0\}$, hence $C \neq \{0\}$.

Let $(e_\iota)_{\iota \in I}$ be a maximal family of mutually orthogonal non-zero projections in C. Then

$$\bigvee_\iota e_\iota = 1_M.$$

Indeed, $e_o = 1_M - \bigvee_\iota e_\iota$ belongs to $C' \cap M$, so Lemma 4 (iv) yields $e_o = \bigvee\{f \in C; f \leq e_o \text{ projection}\}$. Thus $e_o \neq 0$ would imply the existence of some projection $0 \neq f \leq e_o$ in C, contradicting the maximality of $(e_\iota)_{\iota \in I}$.

Denoting by Z the centre of M, we call central partition of 1_M any set of mutually orthogonal projections in Z with least upper bound 1_M. The projections

$$\bigvee_{p \in \mathcal{P}} \Big(\sum_{\iota \in I_p} e_\iota \Big) p\,, \quad \mathcal{P} \text{ central partition of } 1_M\,, \quad I_p \subset I \text{ finite for any } p \in \mathcal{P}$$

belong to $C' \cap M$ and are finite (see [Be],§15, Proposition 8), hence they belong to , $C' \cap A = C$. We show that their family is an (increasing positive) approximate unit for A. For we have to prove that every finite projection e in M has the property

$$\text{(P)} \qquad \begin{cases} \text{for every } \varepsilon > 0 \text{ there are } \mathcal{P} \text{ and } I_p\,, p \in \mathcal{P}, \text{ with} \\ \Big\| \Big(1_M - \bigvee\limits_{p \in \mathcal{P}} \big(\sum\limits_{\iota \in I_p} e_\iota \big) p \Big) e \Big\| \leq \varepsilon\,. \end{cases}$$

But standard arguments show that every finite projection e in M is of the form

$$e = \bigvee_{n \geq 1} (e_{n,1} + \cdots + e_{n,n}) p_n\,,$$

where $p_n, n \geq 1$ are mutually orthogonal projections in Z and, for every $n \geq 1, e_{n,1}, \ldots, e_{n,n}$ are mutually orthogonal abelian projections of central support p_n (use [Be], §18, Exercises 3, 4 and Proposition 1), so it is enough to prove (P) for every abelian projection e in M. Moreover, since every abelian projection is majorized by an abelian projection of central support 1_M, without loss of generality we can restrict us to the case of an abelian projection e of central support 1_M.

For every $x \in M$ there exists a unique $\Phi_e(x) \in Z$ such that

$$exe = \Phi_e(x) e$$

(see [Be], §15, Proposition 6 and §5). Clearly, $\Phi_e : M \to Z$ is a conditional expectation and, according to [Kap 2], Lemma 7, it is completely additive on the projection lattice of M. Furthermore, $Z \ni z \mapsto ze \in Ze$ being $*$-isomorphism, we have

$$\|xe\|^2 = \|ex^*xe\| = \|\Phi_e(x^*x)e\| = \|\Phi_e(x^*x)\|, \quad x \in M\,.$$

Now, by the complete additivity of Φ_e,

$$\sum_\iota \Phi_e(e_\iota) = \Phi_e(1_M) = 1_M\,.$$

Thus, according to [Kap 2], Lemma 5, for every $e > 0$ there exist a central partition $\mathcal{P}$ of 1_M and finite sets $I_p \subset I, p \in \mathcal{P}$ such that

$$\Big\|\Big(1_M - \sum_{\iota\in I_p}\Phi_e(e_\iota)\Big)p\Big\| \leq \varepsilon^2 \text{ for all } p \in \mathcal{P}\,.$$

But then we have for every $p \in \mathcal{P}$

$$\Big\|\Big(1_M - \sum_{\iota\in I_p} e_\iota\Big)pe\Big\|^2 = \Big\|\Phi_e((1_M - \sum_{\iota\in Ip} e_\iota\Big)p\Big\| =$$

$$= \Big\|\Big(1_M - \sum_{\iota\in Ip}\Phi_e(e_\iota)\Big)p\Big\| \leq \varepsilon^2,$$

so, taking into account [Kap 1], Lemma 2.5,

$$\Big\|\Big(1_M - \bigvee_{p\in\mathcal{P}}(\sum_{\iota\in I_p} e_\iota)p\Big)e\Big\| = \sup_{p\in\mathcal{P}}\Big\|\Big(1_M - \sum_{\iota\in I_p} e_\iota\Big)pe\Big\| \leq \varepsilon\,.$$

□

2 Abelian Strict Closure in Continuous AW*-algebras

For the treatment of the case of continuous M we need several lemmas on AW^*-algebras , which could be of interest for themselves. First we extend [Z], Lemma 2.2, concerning a Darboux property of normal functionals on von Neumann algebras without minimal projections, to the case of centre valued completely additive maps on the projection lattice of a continuous AW^*-algebra (similar results can be found in [Ars-Z] and, for tracial maps, in [Kad 2], Prop. 3.13, [Kaf], Prop. 27).

Lemma 2. *Let M be a continuous AW^*-algebra, Z its centre, C a masa of M, and*

$$\Phi : \{e \in M; e \text{ projection }\} \to Z^+$$

a completely additive map such that

$$\Phi(ep) = \Phi(e)p, \quad e \in M \text{ and } p \in Z \text{ projections.}$$

Then, for every projection $e \in C$,

$$\{z \in Z; 0 \leq z \leq \Phi(e)\} = \{\Phi(f); e \geq f \in C \text{ projection}\}\,.$$

Proof. a) First we prove that for every projection $0 \neq g \in C$ there exists a projection $0 \neq h \leq g$ in C such that

$$\Phi(h) \leq \tfrac{1}{2}\Phi(g)\,.$$

The case $\Phi(g) = 0$ being trivial, we can assume without loss of generality that $\Phi(g) \neq 0$.

Let $(g_\iota)_\iota$ be a maximal family of mutually orthogonal projections in Cg such that $\Phi(g_\iota) = 0$ for every ι. Put $g_1 = g - \bigvee_\iota g_\iota \in C$. Then

$$\Phi(g_1) = \Phi(g) - \sum_\iota \Phi(g_\iota) = \Phi(g) \neq 0\,,$$

so $g_1 \neq 0$. By the maximality of $(g_\iota)_\iota$, for no projection $0 \neq g' \leq g_1$ in C can hold $\Phi(g') = 0$.

Now there exists a projection $g_2 \leq g_1$ in C such that $g_2 \notin Z g_1$. For let us assume the contrary, that is that

$$C = Z g_1 + C(1_M - g_1)\,.$$

There exist projections $h_1, h_2 \in M$ such that $g_1 = h_1 + h_2$ and $h_1 \sim h_2$ ([Be], §19, Th. 1) and then

$$C \subset Z h_1 + Z h_2 + C(1_M - g_1)$$

and the maximal abelianness of C imply that

$$C = Z h_1 + Z h_2 + C(1_M - g_1)\,.$$

Thus

$$h_1, h_2 \in C g_1 = Z g_1\,.$$

But, denoting by $z(g_1)$ the central support of g_1,

$$Z z(g_1) \ni z \mapsto z g_1 \in Z g_1\,.$$

is a $*$-isomorphisms and it follows that h_1 and h_2 have orthogonal central supports, in contradiction to $h_1 \sim h_2 \neq 0$.

We claim that $\Phi(g_2)\Phi(g_1 - g_2) \neq 0$. Indeed, otherwise it would exist a projection $p \in Z$ such that

$$\Phi(g_2) = \Phi(g_2) p \text{ and } \Phi(g_1 - g_2) p = 0$$

and it would follow successively

$$\begin{aligned} \Phi(g_2(1_M - p)) = 0 &\text{ and } \Phi((g_1 - g_2) p) = 0\,, \\ (g_2(1_M - p)) = 0 &\text{ and } (g_1 - g_2) p = 0\,, \\ g_2 = g_2 p = g_1 p &\in Z g_1\,. \end{aligned}$$

Let $q \in Z$ denote the support projection of $(\Phi(g_1) - 2\Phi(g_2))_+$. Then

$$\begin{aligned} \Phi(g_1 q) - 2\Phi(g_2 q) &= (\Phi(g_1) - 2\Phi(g_2))_+ \geq 0, \\ \Phi(g_2 q) \leq \frac{1}{2}\Phi(g_1 q) &\leq \frac{1}{2}\Phi(g_1) \leq \frac{1}{2}\Phi(g)\,. \end{aligned}$$

Similarly,

$$\Phi((g_1 - g_2)(1_M - q)) \leq \frac{1}{2}\Phi(g)\,.$$

But we can not have simultaneously

$$\Phi(g_2 q) = 0 \text{ and } \Phi((g_1 - g_2)(1_M - q)) = 0\,,$$

because this would imply

$$\Phi(g_2)\Phi(g_1 - g_2) = \Phi(g_2 q)\Phi(g_1 - g_2) + \Phi(g_2)\Phi((1_M - q)(g_1 - g_2)) = 0\,.$$

Therefore, putting $h = g_2 q$ if $\Phi(g_2 q) \neq 0$ and $h = (g_1 - g_2)(1_M - q)$ otherwise, h is a non-zero projection in C, majorized by g, such that $\Phi(h) \leq \frac{1}{2}\Phi(g)$.

b) Now let $e \in C$ be a projection and let $x \in Z$, $0 \leq z \leq \Phi(e)$ be arbitrary. Choose a maximal family $(f_\iota)_\iota$ of mutually orthogonal projections in Ce satisfying

$$\sum_\iota \Phi(f_\iota) \leq z\,.$$

Then the projection $f = \bigvee_\iota f_\iota \leq e$ belongs to C and

$$\Phi(f) = \sum_\iota \Phi(f_\iota) \leq z\,.$$

We claim that actually $\Phi(f) = z$.

For let us assume the contrary. Then there exist a projection $0 \neq p \in Z$ and $\varepsilon > 0$ such that

$$(z - \Phi(f))p \geq \varepsilon p\,.$$

The projection $g = (e - f)p \in C$ is not zero, because otherwise it would follow

$$0 = (\Phi(e) - \Phi(f))p \geq (z - \Phi(f))p \geq \varepsilon p\,,$$

contradicting $p \neq 0, \varepsilon > 0$. Choosing an integer $n \geq 1$ with $2^{-n}\|\Phi(e - f)\| \leq \varepsilon$, n-fold application of a) yields the existence of a projection $0 \neq h \leq g$ in C such that

$$\Phi(h) \leq 2^{-n}\Phi((e - f)p) \leq \varepsilon p\,.$$

Since $0 \neq h \in Ce$ is orthogonal to every f_ι and

$$\Phi(h) + \sum_\iota \Phi(f_\iota) = \Phi(h) + \Phi(f) \leq \varepsilon p + \Phi(f) \leq z\,,$$

the maximality of $(f_\iota)_\iota$ is contradicted.

□

It is well known that if the projection family $(e_\iota)_\iota$ in a finite AW^*-algebra M is upward directed and, for some projection $f \in M$, $e_\iota \prec f$ for all ι , then $\bigvee_\iota e_\iota \prec f$ (see [Be], §33, Exercise 1). The above statement actually holds in any AW^*-algebra M under the only assumption of the finiteness of f (see Appendix, Cor. 1). Here we give a proof for this, assuming additionally that the projections e_ι are the finite partial sums of a family of mutually orthogonal projections in M :

Lemma 3. *Let M be an AW^*-algebra, $f \in M$ a finite projection, and $(e_\iota)_{\iota \in I}$ a family of mutually orthogonal projections in M such that*

$$\sum_{\iota \in F} e_\iota \prec f \text{ for every finite } F \subset I\,.$$

Then

$$\bigvee_{\iota \in I} e_\iota \prec f\,.$$

Proof According to the theory of Murray-von Neumann equivalence for projections in AW^*-algebras, we can assume without loss of generality that either fMf is of type I_n for some natural number $n \geq 1$, or that it is continuous (see [Be], §15, Th.1, §18, Th. 2, §6, Cor. 2 of Prop. 4).

Let us first assume that fMf is of type I_n. By the Zorn Lemma there exists a maximal set $\mathcal{P}$ of mutually orthogonal central projections in M such that

$$\text{card}\,\{\iota \in I ; pe_\iota \neq 0\} \leq n \text{ for every } p \in \mathcal{P}\,.$$

We claim that $\bigvee \mathcal{P} = 1_M$. For let us assume that $p_o = \bigvee \mathcal{P} \neq 1_M$. Then we can find recursively $n + 1$ indices $\iota_1, \ldots, \iota_{n+1} \in I$ such that

$$p_1 = (1_M - p_o) z(e_{\iota_1}) \ldots z(e_{\iota_{n+1}}) \neq 0\,,$$

where $z(e_\iota)$ denotes the central support of e_ι . By the assumption of the lemma there exist mutually orthogonal projections $f_{\iota_1}, \ldots, f_{\iota_{n+1}} \leq f$ in M such that $e_{\iota_j} \sim f_{\iota_j}$ for every $1 \leq j \leq n+1$. For every $1 \leq j \leq n+1$, the central support of $p_1 f_{\iota_j}$ is p_1 , so there exists an abelian projection $g_j \leq p_1 f_{\iota_j}$ of central support p_1 (see [Be], §18, exercise 4). But then $g_1, \ldots, g_{n+1}$ are mutually orthogonal, equivalent, non-zero projections in fMf (see [Be], §18, Prop.1), which contradicts [Be], §18, Prop. 4.

By the very orthogonal additivity of equivalence in AW^*-algebras (see [Be], §11, Prop. 2) we conclude that

$$\bigvee_{\iota \in I} e_\iota = \bigvee \Big\{ \sum_{pe_\iota \neq 0} p e_{\iota_j}\,;\, p \in \mathcal{P} \Big\} \prec \bigvee \{ pf\,;\, p \in \mathcal{P} \} = f\,.$$

Let us next assume that fMf is continuous and let $x \mapsto x^\natural$ denote the centre valued dimension function of the finite AW^*-algebra fMf (see [Be], Ch.6).

For every $\iota \in I$ there exists a projection $e'_\iota \leq f$ in M such that $e_\iota \sim e'_\iota$. Since $(e'_\iota)^\natural$ does not depend on the choice of e'_ι , we can put

$$e_\iota^\natural = (e'_\iota)^\natural\,.$$

By the assumption of the lemma, for every finite $F \subset I$ we can choose the projections e'_ι , $\iota \in F$, mutually orthogonal and then

$$\sum_{\iota \in F} e_\iota^\natural = \sum_{\iota \in F} (e'_\iota)^\natural = \Big(\sum_{\iota \in F} e'_\iota \Big)^\natural \leq f\,.$$

It follows that all sums

$$\sum_{\iota \in J} e_\iota^\natural \leq f, \quad J \subset I$$

exist in the monotone complete centre of fMf .

Now let us consider the set of all families of mutually orthogonal projections in fMf

$$(f_\iota)_{\iota \in J} \text{ with } J \subset I\,,$$

for which $f_\iota \sim e_\iota$ for every $\iota \in J$. We can endowe this set with the partial order

$$(f_\iota)_{\iota \in J} \leq (f'_\iota)_{\iota \in J'} \iff J \subset J' \text{ and } f_\iota = f'_\iota \text{ for all } \iota \in J\,.$$

By the Zorn lemma there exists a maximal element $(f_\iota)_{\iota \in J}$ of the above partially ordered set. We claim that then $J = I$. For let us assume the existence of some $\iota_o \in I \setminus J$. Since

$$e_{\iota_o}^\natural + \Big(\bigvee_{\iota \in J} f_\iota \Big)^\natural = e_{\iota_o}^\natural + \sum_{\iota \in J} f_\iota^\natural \leq \sum_{\iota \in I} e_\iota^\natural \leq f\,,$$

that is

$$e_{\iota_o}^\natural \leq \Big(f - \bigvee_{\iota \in J} f_\iota \Big)^\natural,$$

by [Be], §33, Th.3 (particular case of the above Lemma 5) there exists a projection $f_{\iota_o} \leq f - \bigvee_{\iota \in J} f_\iota$ in M such that $f_{\iota_o}^\natural = e_{\iota_o}^\natural = \big(e'_{\iota_o}\big)^\natural$, hence $f_{\iota_o} \sim e'_{\iota_o} \sim e_{\iota_o}$. But this contradicts the maximality of $(f_\iota)_{\iota \in J}$.

By the general additivity of equivalence in AW^*-algebras (see [Be], §20, Th. 1) we can conclude also in this case that

$$\bigvee_{\iota \in I} e_\iota \sim \bigvee_{\iota \in I} f_\iota \leq f \,.$$

□

Let M be a semi-finite AW^*-algebra, and A the norm-closed linear span of all finite projections of M. We recall that then $M = M(A)$.

Let us call a masa $\tilde{C}$ of M M-semi-finite if $\tilde{C} \cap A$ is an essential ideal of $\tilde{C}$ or, equivalently, if every non-zero projections in $\tilde{C}$ majorizes a non-zero projection in $\tilde{C} \cap A$ (cf. with [Kaf]. Def. 1). For $\tilde{C} \subset M$ are equivalent:

1) $\tilde{C}$ is an M-semi-finite masa of M;

2) $\tilde{C} = C' \cap M$ for some masa C of A.

Indeed, 2) implies 1) by Lemma 1 (iv), while 1) $\Rightarrow$ 2) follows by noticing that, according to the M-semifiniteness of $\tilde{C}$, every projection in $\tilde{C}$ is the least upper bound of a family of mutually orthogonal projections from $C = \tilde{C} \cap A$, and so $C' \cap M = \tilde{C}' \cap M = \tilde{C}$, $C' \cap A = (C' \cap M) \cap A = \tilde{C} \cap A = C$.

The following result extends [Kad 2], Th. 3.18 and [Kaf], Cor. 31 in the case of an M-semifinite masa :

Theorem 2 (on labeling Murray-von Neumann equivalence classes)**.** *Let M be a semi-finite AW^*-algebra, A the norm closed linear span of all finite projections of M, and C a masa of A. Then*

(i) *for any projections $M \ni f \leq e \in C' \cap M$ there exists a projection $f \sim g \leq e$ in $C' \cap M$;*

(ii) *for any projections $M \ni f \leq e \in C' \cap M$ of equal central supports, f finite and e properly infinite, there is a set $\mathcal{P}$ of mutually orthogonal central projections in M with $\bigvee \mathcal{P} = 1_M$ such that, for every $p \in \mathcal{P}$, ep is the least upper bound in the projection lattice of M of some family of mutually orthogonal projections from C, each one of which is equivalent in M to fp.*

Proof. (a) First we prove (i) in the case $e \in C$. Similarly as in the proof of Lemma 3, we can assume without loss of generality that either $eMe = eAe$ is of type I_n for some natural number $n \geq 1$, or it is continuous.

If eMe is of type I_n, by [Kad 2], Lemma 3.7 there exist mutually orthogonal projections $e_1, \ldots, e_n \in C$ with $\sum\limits_{j=1}^{n} e_j = e$, such that each e_j is abelian in M and has the same central support in M as e (actually [Kad 2], Lemma 3.7 is proved only for von Neumann algebras, but an inspection of the proof shows that it works without any change also in the realm of the AW^*-algebras). On the other hand, using [Be], §18, Exercise 4 and Prop. 4, it is easy to see that there exist mutually orthogonal abelian projections $f_1, \ldots, f_n \in M$ with $\sum\limits_{j=1}^{n} f_j = f$ and central supports $z(f) = z(f_1) \geq \cdots \geq z(f_n)$. By [Be], §18, Prop. 1 it follows that $f_j \sim e_j z(f_j)$ for all $1 \leq j \leq n$, so f is equivalent to $C \ni \sum\limits_{j=1}^{n} e_j z(f_j) \leq e$.

Now let us assume that eMe is continuous and let $x \mapsto x^\natural$ denote the centre valued dimension function of finite AW^*-algebra eMe. Then Lemma 2 yields the existence of a projection $C \ni g \leq e$ such that $g^\natural = f^\natural$, hence $g \sim f$.

(b) Next we prove (i) in the case $f \in A$.

By Lemma 1 (iv) there exists a family $(e_\iota)_{\iota \in I}$ of mutually orthogonal projections in C such that

$$e = \bigvee_{\iota \in I} e_\iota .$$

Let $\mathcal{P}$ be a maximal set of mutually orthogonal central projections in M such that, for every $p \in \mathcal{P}$, there is a finite set $F_p \subset I$ with

$$fp \prec p \sum_{\iota \in F_p} e_\iota \in C .$$

By the above part (a) of the proof, for every $p \in \mathcal{P}$ there exists a projection

$$g(p) \in C \text{ with } fp \sim g(p) \leq p \sum_{\iota \in F_p} e_\iota .$$

If $\bigvee \mathcal{P} = 1_M$ then $f = \bigvee \{fp\,;\, p \in \mathcal{P}\}$ is equivalent to $C' \cap M \ni \bigvee \{g(p)\,;\, p \in \mathcal{P}\} \leq e$, so let us assume in the sequel that $p_o = 1_M - \bigvee \mathcal{P} \neq 0$.

By the maximality of $\mathcal{P}$ and by the comparison theorem (see [Be], §14, Cor. 1 of Prop. 7) we have

$$p_o \sum_{\iota \in F} e_\iota \prec f \text{ for every finite } F \subset I .$$

According to Lemma 3 it follows that

$$p_o e = \bigvee_{\iota \in I} p_o e_\iota \prec f ,$$

so by the Schröder-Bernstein theorem (see [Be],§12) we have

$$f p_o \sim e p_o .$$

Consequently $f = f p_o + \bigvee \{fp\,;\, p \in \mathcal{P}\}$ is equivalent to

$$C' \cap M \ni e p_o + \bigvee \{g(p)\,;\, p \in \mathcal{P}\} \leq e .$$

(c) Now we prove (ii).

Let $\mathcal{P}$ be a maximal set of mutually orthogonal central projections in M such that, for every $p \in \mathcal{P}$, ep is the least upper bound in the projection lattice of M of some family of mutually orthogonal projections from C, each one of which is equivalent in M to fp. We claim that then $\bigvee \mathcal{P} = 1_M$.

For let us assume that $p_o = 1_M - \bigvee \mathcal{P} \neq 0$. We notice that $fp \neq 0$ for any central projection $0 \neq p \leq p_o$ in M : indeed, otherwise p would be orthogonal to the common central support of f and e, so $ep = 0$ would be equal to $fp = 0 \in C$, in contradiction with the maximality of $\mathcal{P}$.

Let $(e_\iota)_{\iota \in I}$ be a maximal family of mutually orthogonal projections in C such that $f p_o \sim e_\iota \leq e p_o$ for all $\iota \in I$. By the comparison theorem there exists a central projection $p_1 \leq p_o$ in M such that

$$\Big(e p_o - \bigvee_{\iota \in I} e_\iota \Big) p_1 \prec f p_1 ,$$

$$\Big(e p_o - \bigvee_{\iota \in I} e_\iota \Big) (p_o - p_1) \succ f (p_o - p_1) .$$

Then $p_1 \neq 0$: indeed, $p_1 = 0$ would imply

$$A \ni f p_o \prec e p_o - \bigvee_{\iota \in I} e_\iota \in C' \cap M$$

and, by the above proved (b), it would exist a projection $fp_o \sim e' \leq ep_o - \bigvee_{\iota\in I} e_\iota$ in $(C' \cap M) \cap A = C$, contradicting the maximality of $(e_\iota)_{\iota\in I}$. Put

$$e_o = ep_1 - \bigvee_{\iota\in I} e_\iota p_1 \prec fp_1 .$$

Then e_o is finite and belongs to $C' \cap M$, so it belongs to $C' \cap A = C$. On the other hand, the proper infiniteness of e and $ep_1 \neq 0$ imply that $ep_1 = e_o + \bigvee_{\iota\in I} e_\iota p_1$ is properly infinite. It follows that the set I is necessarily infinite, hence containing an infinite sequence $\iota_1, \iota_2, \ldots$.

For every $j \geq 1$, $e_o \prec fp_1 \sim e_{\iota_j} p_1 \in C$ and the above proved a) yield the existence of some projection $e_o \sim e^{(1)}_{\iota_j} \leq e_{\iota_j} p_1$ in C. In particular, all projections $e^{(1)}_{\iota_j}$ are equivalent, hence, the projections $e_\iota p_1$ being finite, the projections $e^{(2)}_{\iota_j} = e_{\iota_j} p_1 - e^{(1)}_{\iota_j}$ are also all equivalent (see [Be], §17, Exercise 3). Consequently, the projections from C

$$e'_{\iota_1} = e_o + e^{(2)}_{\iota_1} \text{ and } e'_{\iota_j} = e^{(1)}_{\iota_{j-1}} + e^{(2)}_{\iota_j}, \quad j \geq 2$$

are all equivalent in M to $e^{(1)}_{\iota_1} + e^{(2)}_{\iota_1} = e_{\iota_1} p_1 \sim fp_1$. Clearly, they are mutually orthogonal and

$$\bigvee_{j\geq 1} e'_{\iota_j} = e_o \vee \bigvee_{j\geq 1} e^{(1)}_{\iota_j} \vee \bigvee_{j\geq 1} e^{(2)}_{\iota_j} = e_o \vee \bigvee_{j\geq 1} e_{\iota_j} p_1 .$$

Letting

$$e'_\iota = e_\iota p_1 \text{ for } \iota \in I \setminus \{\iota_1, \iota_2, \ldots\},$$

we conclude that all projections e'_ι, $\iota \in I$, belong to C and are equivalent in M to fp_1. Moreover, they are mutually orthogonal and

$$\bigvee_{\iota\in I} e'_\iota = \bigvee_{j\geq 1} e'_{\iota_j} \vee \bigvee_{\iota\neq\iota_j} e'_\iota = e_o \vee \bigvee_{j\geq 1} e_{\iota_j} p_1 \vee \bigvee_{\iota\neq\iota_j} e_\iota p_1 = e_o \vee \bigvee_{\iota\in I} e_\iota p_1 = ep_1 .$$

But this contradicts the maximality of $\mathcal{P}$.

(d) Finally we prove (i) in full generality.

We can assume without loss of generality that either f is finite, or it is properly infinite. The case of finite f was already settled in (b), so it remains to consider only the case of properly infinite f.

Choose some finite projection $M \ni f_o \leq f$ of the same central support as f (see [Be], §17, Exercise 19 iii)). According to the above proved (c), we can assume without loss of generality that there are families $(e_\iota)_{\iota\in I}$ and $(f_\kappa)_{\kappa\in K}$ of mutually orthogonal projections in M such that

$$e_\iota \sim f_o \sim f_\kappa \text{ for all } \iota \in I \text{ and } \kappa \in K$$

$$\bigvee_{\iota\in I} e_\iota = e, \qquad \bigvee_{\kappa\in K} f_\kappa = f .$$

If card $K \leq$ card I, that is if there exists an injective map $K \ni \kappa \mapsto \iota(\kappa) \in I$, then the projection $g = \bigvee_{\kappa\in K} e_{\iota(\kappa)} \leq e$ belongs to $C' \cap M$ and is equivalent to $\bigvee_{\kappa\in K} f_\kappa = f$. On the other hand, if card $I \leq$ card K, then $e = \bigvee_{\iota\in I} e_\iota \prec \bigvee_{\kappa\in K} f_\kappa = f \leq e$ and the Schröder-Bernstein theorem imply that $e \sim f$.

□

Let us now prove the statement of [Kad 2], Th. 3.18 and [Kaf], Cor. 31 in the case of an M-semifinite masa of an arbitrary semifinite AW^*-algebra M:

Corollary. *Let M be a semifinite AW^*-algebra, A the norm-closed linear span of all finite projections of M, and C a masa of A. If $e \in C' \cap M$ is a projection and $1 \leq n \leq \aleph_o$ is a cardinal number such that e is the least upper bound of n mutually orthogonal, equivalent projections from M, then there exist n mutually orthogonal projections in $C' \cap M$, all equivalent in M, whose least upper bound is e.*

Proof. It is enough to treat separately the case of finite respectively properly infinite e. If e is finite, n can be only a natural number. Let $f_1, \ldots f_n$ be mutually orthogonal, equivalent projections in M with $\sum_{j=1}^{n} f_j = e$. By (i) in the above theorem there exists a projection $f_1 \sim e_1 \leq e$ in C. Since e is finite, it follows that $\sum_{j=2}^{n} f_j \sim e - e_1$, so we can apply again (i) in the above theorem to get a projection $f_2 \sim e_2 \leq e - e_1$ in C. By induction we obtain n mutually orthogonal projections $e_1, \ldots, e_n \in C$ such that $f_j \sim e_j$ for all j and $\sum_{j=1}^{n} e_j = e$.

Now let us assume that e is properly infinite and consider a set I of cardinality n. Choosing a finite projection $M \ni f \leq e$ of the same central support as e (see [Be], §17, Exercise 19 iii)), (ii) in the above theorem entails the existence of a set $\mathcal{P}$ of mutually orthogonal central projections in M with $\bigvee \mathcal{P} = 1_M$ such that, for every $p \in \mathcal{P}$, ep is the least upper bound of some set $\mathcal{E}_p$ of mutually orthogonal projections from C, each one of which is equivalent in M to fp. If $ep \neq 0$ then $\mathcal{E}_p$ must be infinite, so there exists a partition $(\mathcal{E}_{p,\iota})_{\iota \in I}$ of $\mathcal{E}_p$ in n sets of equal cardinality. Then the projections $e_\iota = \bigvee_{ep \neq 0} \bigvee \mathcal{E}_{p,\iota}$, $\iota \in I$, belong to $C' \cap M$, are mutually orthogonal and equivalent in M, and $\bigvee_{\iota \in I} e_\iota = e$.

□

Let M be a finite AW^*-algebra with centre Z and let $x \mapsto x^\natural$ denote its centre valued dimension function (see [Be], Ch. 6). It is known (see [Bl-Ha], II, 1) that $\natural$ can be uniquely extended to a centre valued quasitrace on M, that is to a map $\Phi : M \to Z$ such that

- Φ is linear on commutative $*$-subalgebras of M,
- $\Phi(a + ib) = \Phi(a) + i\Phi(b)$ for all selfadjoint $a, b \in M$,
- Φ acts identically on Z,
- $0 \leqslant \Phi(x^*x) = \Phi(xx^*)$ for all $x \in M$,

and then

- $\Phi(a) \leqslant \Phi(b)$ whenever $a \leqslant b$ are selfadjoint elements of M,
- Φ is norm continuous, more precisely, $\|\Phi(a) - \Phi(b)\| \leqslant \|a - b\|$ for all selfadjoint $a, b \in M$.

We shall use the symbol $\natural$ to denote also the above Φ.

According to classical results of F.J. Murray and J. von Neumann, the centre valued quasitrace of every finite W^*-algebra is additive, hence linear.

It is an open question, raised by I. Kaplansky, whether the centre valued quasitrace of every finite AW^*-algebra is additive. Recently U. Haagerup has proven that the answer to Kaplansky's question is positive for any finite AW^*-algebra which is generated (as an AW^*-algebra) by an exact C^*-subalgebra (see [Haa], Th. 5.11, Prop. 3.12 and Lemma 3.7 (4)).

We notice that if M is a finite AW^*-algebra and $n \geq 1$ is an integer, then the $*$-algebra $\mathrm{Mat}_n(M)$ of all $n \times n$ matrices over M is again a finite AW^*-algebra (see

[Be], §62). Denoting by $\natural$ and $\natural_n$ the respective centre valued quasitraces, it is easily seen that

$$n\cdot\begin{pmatrix} x & 0 & & 0\\ 0 & 0 & & \\ & & \ddots & \\ 0 & & & 0\end{pmatrix}^{\natural_n} = \begin{pmatrix} x^\natural & 0 & & 0\\ 0 & x^\natural & & \\ & & \ddots & \\ 0 & & & x^\natural\end{pmatrix}, \qquad x\in M\,.$$

Moreover the additivity of $\natural$ is equivalent with the validity of

$$2\cdot\begin{pmatrix} x_{11} & x_{12}\\ x_{21} & x_{22}\end{pmatrix}^{\natural_2} = \begin{pmatrix} x_{11}^\natural + x_{22}^\natural & 0\\ 0 & x_{11}^\natural + x_{22}^\natural\end{pmatrix}.$$

Indeed, using the above equality, we get for all $0\leq a,b\in M$

$$\begin{aligned}\begin{pmatrix} (a+b)^\natural & 0\\ 0 & (a+b)^\natural\end{pmatrix} &= 2\cdot\begin{pmatrix} a+b & 0\\ 0 & 0\end{pmatrix}^{\natural_2} =\\ &= 2\cdot\left[\begin{pmatrix} a^{1/2} & b^{1/2}\\ 0 & 0\end{pmatrix}\begin{pmatrix} a^{1/2} & 0\\ b^{1/2} & 0\end{pmatrix}\right]^{\natural_2} =\\ &= 2\cdot\left[\begin{pmatrix} a^{1/2} & 0\\ b^{1/2} & 0\end{pmatrix}\begin{pmatrix} a^{1/2} & b^{1/2}\\ 0 & 0\end{pmatrix}\right]^{\natural_2} =\\ &= 2\cdot\begin{pmatrix} a & a^{1/2}b^{1/2}\\ b^{1/2}a^{1/2} & b\end{pmatrix}^{\natural_2} =\\ &= \begin{pmatrix} a^\natural + b^\natural & 0\\ 0 & a^\natural + b^\natural\end{pmatrix}.\end{aligned}$$

Conversely, assuming that $\natural$ is additive, it is easy to verify that

$$\begin{pmatrix} x_{11} & x_{12}\\ x_{21} & x_{22}\end{pmatrix} \mapsto \frac{1}{2}\begin{pmatrix} x_{11}^\natural + x_{22}^\natural & 0\\ 0 & x_{11}^\natural + x_{22}^\natural\end{pmatrix}$$

is a centre valued quasitrace on $\mathrm{Mat}_2(M)$.

For a given $\delta>0$, we say that the centre valued quasitrace $\natural$ of a finite AW^*-algebra M is δ-subadditive (resp. δ-superadditive) if the map $M_+\ni a\mapsto (a^\natural)^\delta$ is subadditive (resp. superadditive). Clearly, δ-subadditivity (δ-superadditivity) of $\natural$ implies its δ'-subadditivity (δ'-superaddivity) whenever $\delta'<\delta(\delta'>\delta)$. It was proven by U. Haagerup that $\natural$ is always $\frac{1}{2}$-subadditive (see [Haa], Lemma 3.5 (1)) and it seems reasonable to conjecture that it is also always 2-superadditive (or, at least, k-superadditive for some $k\geqslant 1$).

We notice as a curiosity that, for any two projections p, q in a finite AW^*-algebra M with centre valued quasitrace $\natural$,

$$(p+q)^\natural = p^\natural + q^\natural\,.$$

Indeed, since

$$\begin{pmatrix} p+q & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix} p & \pm q\\ 0 & 0\end{pmatrix}\begin{pmatrix} p & 0\\ \pm q & 0\end{pmatrix},$$

$$\begin{pmatrix} p & \pm pq\\ \pm qp & q\end{pmatrix} = \begin{pmatrix} p & 0\\ \pm q & 0\end{pmatrix}\begin{pmatrix} p & \pm q\\ 0 & 0\end{pmatrix},$$

and $\begin{pmatrix} p & pq\\ qp & q\end{pmatrix}$, $\begin{pmatrix} p & -pq\\ -qp & q\end{pmatrix}$ commute, we have

$$\begin{aligned}\begin{pmatrix} (p+q)^\natural & 0 \\ 0 & (p+q)^\natural \end{pmatrix} &= 2\begin{pmatrix} p+q & 0 \\ 0 & 0 \end{pmatrix}^{\natural_2} = \\ &= \begin{pmatrix} p & pq \\ qp & q \end{pmatrix}^{\natural_2} + \begin{pmatrix} p & -pq \\ -qp & q \end{pmatrix}^{\natural_2} = \\ &= 2\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}^{\natural_2} = \\ &= 2\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}^{\natural_2} + 2\begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}^{\natural_2} = \\ &= \begin{pmatrix} p^\natural & 0 \\ 0 & p^\natural \end{pmatrix} + \begin{pmatrix} q^\natural & 0 \\ 0 & q^\natural \end{pmatrix} = \\ &= \begin{pmatrix} p^\natural + q^\natural & 0 \\ 0 & p^\natural + q^\natural \end{pmatrix} .\end{aligned}$$

This can be deduced also from Haagerup's result, taking to account that the C^*-algebra generated by two projections is of type I, hence nuclear, hence exact.

Lemma 4. *Let M be a finite AW^*-algebra, whose centre valued quasitrace $\natural$ is k-superadditive for some $k \geq 1$. Let further $e_1, \ldots, e_n \in M$ be mutually equivalent projections with $\sum\limits_{j=1}^{n} e_j = 1_M$. Then there exists a projection $e_1 \sim p \in M$ such that, for every projection $f \in \{e_1, \ldots, e_n\}' \cap M$,*

$$f^\natural \geq (1 - \|(1_M - f)p\|^2) n^{\frac{1}{k}-1} 1_M .$$

Proof. Let $v_1, \ldots v_n \in M$ be partial isometries such that

$$v_j^* v_j = e_1, \quad v_j v_j^* = e_j, \quad 1 \leq j \leq n .$$

Since

$$\Big(\frac{1}{\sqrt{n}}\sum_{j=1}^{n} v_j\Big)^* \frac{1}{\sqrt{n}}\sum_{j=1}^{n} v_j = \frac{1}{n}\sum_{j_1,j_2=1}^{n} v_{j_1}^* v_{j_2} = \frac{1}{n}\sum_{j=1}^{n} v_j^* v_j = e_1 ,$$

$p = \dfrac{1}{n}\sum\limits_{j_1,j_2=1}^{n} v_{j_1} v_{j_2}^* = \dfrac{1}{\sqrt{n}}\sum\limits_{j=1}^{n} v_j \Big(\dfrac{1}{\sqrt{n}}\sum\limits_{j=1}^{n} v_j\Big)^*$ is a projection in M equivalent to e_1.

Now let the projection

$$f \in \{e_1, \ldots, e_n\}' \cap M$$

be arbitrary and set $\delta = \|(1_M - f)p\|$. Since the case $\delta = 1$ is trivial, we can assume without loss of generality that $\delta < 1$. Then

$$\|p - pfp\| = \|(1_M - f)p\|^2 = \delta^2 < 1 ,$$

so $pfp \geq (1-\delta^2)p$ is invertible in pMp. Thus the polar decomposition $fp = w \cdot |fp|$ exists in the C^*-algebra generated by p and f and we have

$$w^* w = p, fpf = w(pfp)w^* \geq (1 - \delta^2) w w^* .$$

Let us denote $\zeta = e^{i\frac{n}{\pi}}$. Then

$$u = \sum_{j=1}^{n} \zeta^j e_j \in \{e_1, \dots, e_n, f\}' \cap M$$

is unitary. Since

$$\begin{aligned} u^m p u^{-m} &= \frac{1}{n} \sum_{j,j_1,j_2,j'=1}^{n} \zeta^{mj} e_j v_{j_1} v_{j_2}^* \zeta^{-mj'} e_{j'} = \\ &= \frac{1}{n} \sum_{j_1,j_2=1}^{n} \zeta^{m(j_1-j_2)} v_{j_1} v_{j_2}^* \end{aligned}$$

and

$$\sum_{m=1}^{n} \zeta^{mj} = 0 \text{ for every } 1 \leq j \leq n-1\,,$$

we have

$$\begin{aligned} \sum_{m=1}^{n} u^m p u^{-m} &= \frac{1}{n} \sum_{j_1,j_2=1}^{n} \left(\sum_{m=1}^{n} \zeta^{m(j_1-j_2)} \right) v_{j_1} v_{j_2}^* = \\ &= \frac{1}{n} \sum_{j_1=1}^{n} n v_{j_1} v_{j_1}^* = 1_M \end{aligned}$$

Therefore

$$f = f \sum_{m=1}^{n} u^m p u^{-m} f = \sum_{m=1}^{n} u^m (fpf) u^{-m} \geq (1-\delta^2) \sum_{m=1}^{n} u^m w w^* u^{-m}$$

and, using the superadditivity of $\natural$, we get

$$\begin{aligned} f^\natural &\geq (1-\delta^2) \bigg(\sum_{m=1}^{n} u^m w w^* u^{-m} \bigg)^\natural \geq \\ &\geq (1-\delta^2) \bigg(\sum_{m=1}^{n} \big((u^m w w^* u^{-m})^\natural \big)^k \bigg)^{\frac{1}{k}} = \\ &= (1-\delta^2) \Big(n \big((w^* w)^\natural \big)^k \Big)^{\frac{1}{k}} = (1-\delta^2) n^{\frac{1}{k}} p^\natural \,. \end{aligned}$$

But $p^\natural = e_j^\natural$ for all $1 \leq j \leq n$, so

$$n p^\natural = \sum_{j=1}^{n} e_j^\natural = \Big(\sum_{j=1}^{n} e_j \Big)^\natural = 1_M$$

and we conclude that $f^\natural \geq (1-\delta^2) n^{\frac{1}{k}-1} 1_M$.

□

Now we are ready to prove the following

Theorem 3 (on the abelian strict closure in continuous semi-finite AW^*-algebras)**.** *Let M be a continuous semi-finite AW^*-algebra such that, for some finite projection $e_o \in M$ of central support 1_M and some $k \geq 1$, the centre valued quasitrace of $e_o M e_o$ is k-superadditive. Let further A denote the norm-closed linear span of all finite projections of M, and C a masa of A. Then the strict closure of C in $M = M(A)$ is C.*

Proof. Let us assume that the strict closure $\overline{C}^{\beta} \subset C' \cap M$ of C contains some $0 \leq x \notin C$.

(a) First we prove that then $\overline{C}^{\beta}$ contains some projection $e \notin C$.

For let e_δ denote, for every $\delta > 0$, the support of $(x - \delta 1_M)_+$ in the AW^*-subalgebra $C' \cap M$ of M. Then

$$x e_\delta \geq \delta e_\delta, \quad x(1_M - e_\delta) \leq \delta(1_M - e_\delta).$$

In particular, there exists $0 \leq y \in C' \cap M$ with $yx = e_\delta$. Moreover, $e_\delta \in \overline{C}^{\beta}$. Indeed, by Lemma 1 (i) there is a net $(b_\iota)_\iota$ in C with

$$0 \leq b_\iota \leq x \text{ for all } \iota, \quad b_\iota \to x \text{ strictly.}$$

Then $0 \leq y b_\iota \in C' \cap A = C$ for all ι and

$$\|(e_\delta - y b_\iota)a\| = \|y(x - b_\iota)a\| \leq \|y\| \cdot \|(x - b_\iota)a\| \to 0.$$

for every $a \in A$.

(b) Next we prove the existence of an infinite sequence of mutually orthogonal projections $0 \neq e_1, e_2, \ldots \in C$, all equivalent in M to $e_o q_o$ for some projection q_o in the centre Z of M, such that $\bigvee_{n \geq 1} e_n \in \overline{C}^{\beta}$.

Let e be a projection as in (a). Then e is not finite, so there exists a projection $q \in Z$ such that eq is properly infinite. But then, by the comparison theorem, there exists a projection $0 \neq q_o \in Z$ such that $e_o q_o \prec eq$. Since the central support of e_o is 1_M, we have $q_o \leq q$.

Now, according to (ii) in Theorem 2 (on labeling Murray-von Neumann equivalence classes), there exists a family $(e_\iota)_{\iota \in I}$ of mutually orthogonal projections in C, all equivalent in M to $e_o q_o \neq 0$, such that $\bigvee_{\iota \in I} e_\iota = e q_o$. I must be infinite, so it contains an infinite sequence $\iota_1, \iota_2, \ldots$. Put

$$e_n = e_{\iota_n}, \quad n \geq 1.$$

Then $\bigvee_{n \geq 1} e_n$ belongs to $\overline{C}^{\beta}$. Indeed, since $\bigvee_{n \geq 1} e_n \in C' \cap M$, if $(b_\kappa)_\kappa$ is a net in C which converges strictly to e, then the net $\big(b_\kappa \bigvee_{n \geq 1} e_n\big)_\kappa$ is contained in C and converges clearly to $e \bigvee_{n \geq 1} e_n = \bigvee_{n \geq 1} e_n$ in the strict topology of M.

(c) Finally we prove that the statement in (b) leads to a contradiction.

Let us denote by $\natural$ the map $\bigcup_{n \geq 1} e_n M e_n \to Z q_o$ such that, for every $n \geq 1$, $e_n M e_n \ni x \longmapsto x^\natural e_n$ is the centre valued quasitrace of $e_n M e_n$. It is easy to see that $\natural$ takes the same value in two projections from $\bigcup_{n \geq 1} e_n M e_n$ if and only if they are equivalent in M.

Let $n \geq 1$ be arbitrary and let $j_n = \big[n^{\frac{k+1}{k}}\big] \geq 1$ denote the integer part of $n^{\frac{k+1}{k}}$. According to the corollary of Theorem 2 (on labeling Murray-von Neumann equivalence classes), there exist projections

$$e_{n,1}, \ldots, e_{n,j_n} \in C, \quad \sum_{j=1}^{j_n} e_{n,j} = e_n,$$

such that

$$e_{n,j}^\natural = \frac{1}{j_n} q_o \text{ for all } 1 \leq j \leq j_n.$$

Since $e_n \sim e_o q_o$, the centre valued quasitrace of $e_n M e_n$ is k-superadditive and Lemma 4 yields the existence of a projection $p_n \in e_n M e_n$ with $p_n^\natural = \frac{1}{j_n} q_o$ such that, for every projection $g \in \{e_{n,1}, \ldots, e_{n,j_n}\}' \cap e_n M e_n$,

$$g^\natural \geq \left(1 - \|(e_n - g)p_n\|^2\right) \frac{1}{j_n^{\frac{k-1}{k}}} q_o \geq \left(1 - \|(e_n - g)p_n\|^2\right) \frac{1}{n} q_o .$$

Now put $p = \bigvee_{n \geq 1} p_n$. Since $p_n^\natural = \frac{1}{j_n} q_o$ and $\sum_{n \geq 1} \frac{1}{j_n} < +\infty$, using Lemma 2 it is easy to verify that p is equivalent to a subprojection of the sum of finitely many e_n's. In particular, p is finite, that is $p \in A$. Therefore, $\bigvee_{n \geq 1} e_n$ being in $\overline{C}^\beta$, Lemma 1 (iii) yields the existence of a projection $\bigvee_{n \geq 1} e_n \geq f \in C$ with

$$\left\| \left(\bigvee_{n \geq 1} e_n - f \right) p \right\| \leq \frac{1}{\sqrt{2}} .$$

But then, for every $n \geq 1$, $f e_n$ is a projection in $C \cap e_n M e_n \subset \{e_{n,1}, \ldots, e_{n,j_n}\}' \cap e_n M e_n$ and the aboves yield

$$(f e_n)^\natural \geq \left(1 - \|(e_n - f e_n)p_n\|^2\right) \frac{1}{n} q_o \geq \frac{1}{2n} q_o .$$

Since $\sum_{n \geq 1} \frac{1}{2n} = +\infty$, using again Lemma 2, it is easily seen that $f = \bigvee_{n \geq 1}(f e_n)$ is equivalent to $\bigvee_{n \geq 1} e_n$. In particular, f is properly infinite, in contradiction with $f \in C \subset A$.

$\square$

3 Weyl-von Neumann-Berg-Sikonia type theorems

We recall that any Rickart C^*-algebra M is σ-normal, what means that, for every increasing sequence $(e_k)_{k \geq 1}$ of projections in M, the least upper bound of $(e_k)_{k \geq 1}$ in the projection lattice of M is actually its least upper bound in the ordered space M_h of all self-adjoint elements of M (see [A-Go 2] or [Sa]). Therefore we shall speak in the sequel simply about the least upper bound of increasing sequences of projections in M.

Let us first prove a lemma about the sequential approximability of a projection in a Rickart C^*-algebra from a two-sided ideal :

Lemma 5. *Let M be a unital Rickart C^*-algebra, $\mathcal{J}$ a two-sided ideal of M, and $f \in M$ a projection. Then the following statements are equivalent* :

(a) *there exists a sequence $(b_k)_{k \geq 1}$ of positive elements in $\mathcal{J}$ such that $b_k \leq f$ for all $k \geq 1$ and every projection $e \in M$ with $b_k \leq e$, $k \geq 1$, satisfies $f \leq e$;*

(b) *there exists an increasing sequence $(f_k)_{k \geq 1}$ of projections in $\mathcal{J}$, whose least upper bound in M is f.*

Proof. Let us assume that (a) holds and put

$$f_{k,l} = \text{ support of } \left(b_k - \frac{1}{l} 1_M\right)_+ \leq f , \qquad k, l \geq 1 ,$$

$$f_n = \bigvee_{1 \leq k, l \leq n} f_{k,l} \text{ in the projection lattice of } M \leq f , \qquad n \geq 1 .$$

Since $b_k f_{k,l} \geq \dfrac{1}{l} f_{k,l}$, and so $f_{k,l}$ can be factorized by $b_k \leq f$, we have $f \geq f_{k,l} \in \mathcal{J}$ for all k and l . Further, using the validity of the Parallelogramm Law in all Rickart C^*-algebras (see [Be], §13, Th. 1), we obtain also $f \geq f_n \in \mathcal{J}$, $n \geq 1$.

Now $(f_n)_{n\geq 1}$ is an increasing sequence, whose least upper bound in the projection lattice of M is f . Indeed, if $e \in M$ is a projection which majorizes every f_n , hence every $f_{k,l}$, then we have for all k and l

$$b_k^{\frac{1}{2}}(1_M - e)b_k^{\frac{1}{2}} \leq b_k^{\frac{1}{2}}(1_M - f_{k,l})b_k^{\frac{1}{2}} \leq \frac{1}{l}(1_M - f_{k,l}) ,$$
$$\|(1_M - e)b_k^{\frac{1}{2}}\|^2 \leq \frac{1}{l} .$$

Thus

$$b_k = e\, b_k e \leq e \text{ for all } k \geq 1$$

and it follows that $f \leq e$.

Conversely, (b) obviously implies (a) with $b_k = f_k$.

□

For unital Rickart C^*-algebras we have the following Weyl-von Neumann-Berg-Sikonia type result (cf. with [Z], Theorem 3.1 and [Ak-Ped], §4) :

Theorem 4. *Let M be a unital Rickart C^*-algebra, and $\mathcal{J}$ a norm-closed two-sided ideal of M , which contains a sequence of positive elements such that 1_M is the only projection in M majorizing the sequence. Then, for any normal $y \in M$ and every $\varepsilon > 0$, there exist a masa C of $\mathcal{J}$ and an element x of the masa $C' \cap M$ of M , such that*

1) *C contains an increasing sequence of projections, whose least upper bound in M is 1_M ,*
2) *$y - x \in \mathcal{J}$ and $\|y - x\| \leq \varepsilon$.*

Remark. *We notice that in* Theorem 4 *$C' \cap M$ is the sequentially monotone closure of C in the following sense : every $0 \leq a \in C' \cap M$ is the least upper bound in M_h of some increasing sequence of positive elements from $\mathcal{J}$.*

Indeed, if $\big(e_k\big)_{k\geq 1}$ is an increasing sequence of projections in C , whose least upper bound in M is 1_M , then $\big(a^{1/2}e_k a^{1/2}\big)_{k\geq 1}$ is an increasing sequence of positive elements from $\mathcal{J}$, whose least upper bound in A_h is $a^{1/2}1_M a^{1/2} = a$ (see [S-Z], 9.14, the remark after Proposition 3).

□

For the proof of Theorem 4 we need the next result on quasi-central approximate units, implicitely contained in [Z], Proposition 1.2 :

Lemma 6. *Let M be a unital Rickart C^*-algebra, $\mathcal{J}$ an essential, norm-closed, two-sided ideal of M , and $B \subset M$ a commutative C^*-subalgebra. Then the upward directed set of all projections of $\mathcal{J}$ contains a subnet $(e_\iota)_{\iota\in I}$ which, besides being automatically approximate unit for $\mathcal{J}$, is quasi-central for B , that is*

$$\lim_{\iota} \| e_\iota b - b\, e_\iota\| = 0 \textit{ for all } b \in B .$$

Proof. Passing to the Rickart C^*-subalgebra of M generated by B and 1_M (see e.g. [S-Z], 9.11 (3)), we can assume without loss of generality that B is a Rickart C^*-subalgebra of M containing 1_M .

Let $\mathcal{P}$ denote the set of all finite sets P of projections from B satisfying the equality $\sum_{p\in P} p = 1_M$ and set

$$I = \{ f \in A \,;\, f \text{ projection } \} \times \mathcal{P}\,.$$

We endow I with a partial order by putting $(f_1, P_1) \leq (f_2, P_2)$ whenever $f_1 \leq f_2$ and the C^*-algebra $C^*(P_1)$ generated by P_1 is contained in $C^*(P_2)$ (that is the partition P_2 is a refinement of P_1). Clearly, in this way I becomes an upward directed ordered set.

Let $\iota = (f, P) \in I$ be arbitrary. For every $p \in P$, the right support $\mathbf{r}(fp)$ of fp is equivalent in M to the left support $\mathbf{l}(fp) \leq f \in \mathcal{J}$ (see [A] or [A-Go 1]), so it belongs to $\mathcal{J}$. Thus

$$e_\iota = \sum_{p\in P} \mathbf{r}(fp)$$

is a projection in $\mathcal{J}$. Since every $\mathbf{r}(fp) \leq p$ belongs to the commutant P', also $e_\iota \in P'$. Furthermore,

$$f \leq e_\iota\,.$$

Indeed, for every $q \in P$,

$$fq = fq\,\mathbf{r}(fq) = \sum_{p\in P} fq\,\mathbf{r}(fp) = fq\,e_\iota\,,$$

so

$$f = f\sum_{q\in P} q = \sum_{q\in P} fqe_\iota = fe_\iota \leq e_\iota\,.$$

It is easily seen that

$$\iota_1 \leq \iota_2 \Rightarrow e_{\iota_1} \leq e_{\iota_2}\,,$$

so $(e_\iota)_{\iota\in I}$ is a subnet of the upward directed set of all projections of $\mathcal{J}$.

Now, the upward directed set of all projections f of $\mathcal{J}$ is an increasing approximate unit for $\mathcal{J}$. Indeed, $\big\{x \in \mathcal{J}\,;\, \lim\limits_{f} \|x(1_M - f)\| = 0\big\}$ is a norm-closed linear subspace of $\mathcal{J}$ containing all projections from $\mathcal{J}$, hence it is equal to $\mathcal{J}$. Thus also the subnet $(e_\iota)_{\iota\in I}$ is an approximate unit for $\mathcal{J}$.

On the other hand, the norm-closed linear subspace $\big\{b \in B; \lim\limits_{\iota} \|e_\iota b - be_\iota\| = 0\big\}$ contains every projection from B: for any projection $p \in B$ and every $\iota = (f, P)$ with $p \in C^*(P)$ we have $e_\iota \in P' \cap A = C^*(P)' \cap \mathcal{J}$, so $e_\iota p - pe_\iota = 0$. Consequently the above subspace of B is actually equal to B.

□

Proof of Theorem 4 . Put $y_1 = \dfrac{1}{2}(y + y^*)$, $y_2 = \dfrac{1}{2i}(y - y^*)$ and

$$p_j(\lambda) = \text{ support of } (y_j - \lambda 1_M) \text{ in } M\,, \quad \lambda \in \mathbb{R}\,.$$

Let further $\{\lambda_1, \lambda_2, \dots\}$ be the countable set of all rational numbers. Then

$$a = \sum_{k=1}^{\infty} 3^{-(2k-1)}(2p_1(\lambda_k) - 1_{A^{**}}) + \sum_{k=1}^{\infty} 3^{-2k}(2p_2(\lambda_k) - 1_{A^{**}}) + \frac{1}{2}1_{A^{**}} \in M\,,$$

$$0 \leq a \leq 1_M$$

and it is easy to see that the C^*-subalgebra of M generated by a and 1_M contains all projections $p_j(\lambda)\,,\ j=1,2\,,\ \lambda\in\mathbb{Q}$, hence also $y=y_1+iy_2$. Therefore there exists a continuous function $f:[0,+\infty)\to\mathbb{C}$ such that $y=f(a)$. Furthermore, by a well known continuity property of the functional calculus (see e.g. [S-Z], 1.18 (5)), there exists some $\delta>0$ such that

$$0\le b\in M\,,\ \|a-b\|\le\delta\implies\|f(a)-f(b)\|\le\varepsilon\,.$$

Now, by Lemma 5, there exists an increasing sequence $\big(f_k\big)_{k\ge1}$ of projections in $\mathcal{J}$, whose least upper bound in M is 1_M. Using Lemma 6, we can then construct by induction a sequence $0=e_o\le e_1\le e_2\le\dots$ of projections in $\mathcal{J}$ such that

$$f_k\le e_k\,,\quad\|e_ka-ae_k\|\le2^{-k-1}\delta\,.$$

Since the elements e_k and $(e_k-e_{k-1})a(e_k-e_{k-1})$ of $\mathcal{J}$ are mutually commuting, there exists a masa C of $\mathcal{J}$ containing all of them. Then C contains the increasing projection sequence $(e_k)_{k\ge1}$, whose least upper bound in M is 1_M.

Let us denote

$$\begin{aligned}&b_o=a\,,\\&b_n=\sum_{k=1}^{n}(e_k-e_{k-1})a(e_k-e_{k-1})+(1_M-e_n)a(1_M-e_n)\,,\quad n\ge1\,.\end{aligned}$$

Then, for every $n\ge1$,

$$\begin{aligned}&b_{n-1}-b_n=\\&=(1_M-e_{n-1})\big(a-(1_M-e_n)a(1_M-e_n)-e_nae_n\big)(1_M-e_{n-1})=\\&=(1_M-e_{n-1})\cdot[e_n\,,\,e_na-ae_n]\cdot(1_M-e_{n-1})\,,\end{aligned}$$

$$\|b_{n-1}-b_n\|\le2\|e_na-ae_n\|\le2^{-n}\delta\,.$$

It follows that $\sum\limits_{n=1}^{\infty}\|b_{n-1}-b_n\|\le\delta$, so the sequence $(b_n)_{n\ge1}$ is norm convergent to some $b\in M(A)^+$ and

$$\|a-b\|=\lim_{n\to\infty}\|b_o-b_n\|\le\delta\,.$$

Put $x=f(b)$.

We claim that $b\in C'\cap M$, hence also $x\in C'\cap M$. Since $C'\cap M$ is a masa of M (see Lemma 1 (iv)), it is enough to prove that b is commuting with all elements $a'\in C'\cap M\subset\big\{e_k\,,(e_k-e_{k-1})a(e_k-e_{k-1})\,;\,k\ge1\big\}'\cap M$. For we notice that, for every $n\ge1$,

$$b_na'-a'b_n=(1_M-e_n)(aa'-a'a)(1_M-e_n)\,,$$

hence

$$\begin{aligned}|b_na'-a'b_n|^2&\le(1_M-e_n)|aa'-a'a|^2(1_M-e_n)\le\\&\le\|aa'-a'a\|^2(1_M-e_n)\,.\end{aligned}$$

Therefore

$$|b_na'-a'b_n|^2\le\|aa'-a'a\|^2(1_M-e_k)\,,\quad n\ge k\ge1$$

and, passing to limit for $n\to\infty$, we get for every $k\ge1$

$$|ba'-a'b|^2\le\|aa'-a'a\|^2(1_M-e_k)\,,$$

$$\text{support of } |ba' - a'b|^2 \text{ in } M \text{ is } \leq 1_M - e_k\,.$$

Since the least upper bound of $(e_k)_{k\geq 1}$ in M is 1_M, it follows that $ba' - a'b = 0$.

Finally, according to the choice of δ, $\|a-b\| \leq \delta$ implies that

$$\|y - x\| = \|f(a) - f(b)\| \leq \varepsilon\,.$$

On the other hand,

$$\begin{aligned} a - b_n &= \sum_{k=1}^{n} (b_{k-1} - b_k) = \\ &= \sum_{k=1}^{n} (1_M - e_{k-1}) \cdot [e_k\,,\, e_k a - a e_k] \cdot (1_M - e_{k-1}) \in \mathcal{J} \end{aligned}$$

implies by passing to the limit for $n \to \infty$ that $a - b \in \mathcal{J}$. Using the Weierstrass Approximation Theorem, we infer that $y - x = f(a) - f(b) \in \mathcal{J}$.

$\square$

We shall prove that in Theorem 4 the element x can be found under the form of an "infinite linear combination" of a sequence of mutually orthogonal projections from $\mathcal{J}$. To this aim we need an appropriate understanding of the summation of series in Rickart C^*-algebras.

We recall that every commutative Rickart C^*-algebra C is sequentially monotone complete (see e.g. [S-Z], 9.16, Proposition 1). Thus, if $\big(a_k\big)_{k\geq 1}$ is a sequence in C^+ such that the partial sums $\sum_{k=1}^{n} a_k\,, n \geq 1$, are bounded, then there exists the least upper bound in C_h

$$\sum_{k=1}^{\infty} a_k = \sup\left\{ \sum_{k=1}^{n} a_k\,;\, n \geq 1 \right\} \in C^+\,.$$

Let next M be an arbitrary Rickart C^*-algebra, $\big(a_k\big)_{k\geq 1}$ a bounded sequence in M^+ such that the supports $\mathbf{s}(a_k)\,,\ k \geq 1$, are mutually orthogonal, and $\big(e_k\big)_{k\geq 1}$ a sequence of mutually orthogonal projections in M, for which $\mathbf{s}(a_k) \leq e_k\,,\ k \geq 1$ (we can take, for example, $e_k = \mathbf{s}(a_k)$). Then $\big\{a_k\,;\, k \geq 1\big\} \cup \big\{e_k\,;\, k \geq 1\big\}$ generates a commutative Rickart C^*-subalgebra C of M, so there exists $a = \sum_{k=1}^{\infty} a_k \in C^+$. Moreover, a is the least upper bound of the partial sums $\big\{ \sum_{k=1}^{n} a_k\,;\, n \geq 1\big\}$ even in M_h. Indeed, by the σ-normality of the Rickart C^*-algebras, $\bigvee_{k=1}^{\infty} e_k$ is the least upper bound in M_h of the sequence $\big(\bigvee_{k=1}^{n} e_k\big)_{n\geq 1}$ and it follows that

$$a = a^{1/2}\Big(\bigvee_{k=1}^{\infty} e_k\Big) a^{1/2} \text{ is the least upper bound in } M_h \text{ of}$$

$$\text{the increasing sequence } a^{1/2}\Big(\bigvee_{k=1}^{n} e_k\Big) a^{1/2} = \sum_{k=1}^{n} a_k\,,\ n \geq 1$$

(see [S-Z], 9.14, the remark after Proposition 3). In particular, a is the only element of M_h satisfying the conditions

$$a\, e_k = a_k\,,\ k \geq 1\,, \qquad \mathbf{s}(a) \leq \bigvee_{k=1}^{\infty} e_k\,.$$

For sake of completeness we notice that, by the above characterization, if $\big(e_k\big)_{k\geq 1}$ is a sequence of mutually orthogonal projections in M, then $\sum_{k=1}^{\infty} e_k = \bigvee_{k=1}^{\infty} e_k$.

Now let $\big(x_k\big)_{k\geq 1}$ be a bounded sequence in M such that, denoting by $\mathbf{l}(x_k)$ the left support of x_k and by $\mathbf{r}(x_k)$ the right one, the projections $\mathbf{l}(x_k)\vee\mathbf{r}(x_k)$, $k\geq 1$, are mutually orthogonal. Then we can define

$$\sum_{k=1}^{\infty} x_k = \Big(\sum_{k=1}^{\infty}(\text{Re } x_k)_+ - \sum_{k=1}^{\infty}(\text{Re } x_k)_-\Big) + i\Big(\sum_{k=1}^{\infty}(\text{Im } x_k)_+ - \sum_{k=1}^{\infty}(\text{Im } x_k)_-\Big).$$

It is easy to see that, if $\big(e_k\big)_{k\geq 1}$ is any sequence of mutually orthogonal projections in M such that $\mathbf{l}(x_k)\vee\mathbf{r}(x_k)\leq e_k$, $k\geq 1$, then $\sum_{k=1}^{\infty} x_k$ is the only element $x\in M$, for which

$$(*)\qquad x\,e_k = e_k x = x_k\,,\ k\geq 1\,,\qquad \mathbf{l}(x)\vee\mathbf{r}(x)\leq \bigvee_{k=1}^{\infty} e_k\,.$$

By the aboves, considering the direct product C^*-algebra

$$\bigoplus_{k=1}^{\infty} e_k M e_k = \Big\{ (y_k)_{k\geq 1}\in \prod_{k=1}^{\infty} e_k M e_k\,;\ \sup_{k\geq 1}\|y_k\| < +\infty \Big\},$$

the mapping

$$\bigoplus_{k=1}^{\infty} e_k M e_k \ni (y_k)_{k\geq 1} \longmapsto \sum_{k=1}^{\infty} y_k \in M$$

is well defined and it is an injective $*$-homomorphism. Consequently

$$(**)\qquad \Big\| \sum_{k=1}^{\infty} x_k \Big\| = \sup_{k\geq 1}\|x_k\|\,.$$

Finally, let $\big(e_k\big)_{k\geq 1}$ be a sequence of mutually orthogonal projections in M , and $(x_k)_{k\geq 1}\,,\ (y_k)_{k\geq 1}\in \bigoplus_{k=1}^{\infty} e_k M e_k$. Denoting by $\overline{\text{lin}}\{x_k - y_k\,;\ k\geq 1\}$ the norm-closed linear subspace of M generated by $\{x_k - y_k\,;\ k\geq 1\}$, we have

$$(***)\qquad \sum_{k=1}^{\infty} x_k - \sum_{k=1}^{\infty} y_k \in \overline{\text{lin}}\{x_k - y_k\,;\ k\geq 1\}\ \text{ if }\ \|x_k - y_k\| \longrightarrow 0\,.$$

Indeed, according to (**), we have :

$$\Big\| \sum_{k=1}^{\infty} x_k - \sum_{k=1}^{\infty} y_k - \underbrace{\sum_{k=1}^{n}(x_k - y_k)}_{\in\, \overline{\text{lin}}\{x_k - y_k\,;\ k\geq 1\}} \Big\| = \sup_{k\geq n+1}\|x_k - y_k\| \xrightarrow{n\to\infty} 0\,.$$

A slight modification of the proof of Theorem 4 yields the following Weyl-von Neumann-Berg-Sikonia type result, which is much closer to [Z], Theorem 3.1 than Theorem 4 :

Theorem 5. *Let M be a unital Rickart C^*-algebra, and $\mathcal{J}$ a norm-closed two-sided ideal of M, which contains a sequence of positive elements such that 1_M is the only projection in M majorizing the sequence. Then, for any normal $y\in M$ and every $\varepsilon > 0$, there are*

- *a sequence $(p_k)_{k\geq 1}$ of mutually orthogonal projections in $\mathcal{J}$,*
- *a sequence $(\lambda_k)_{k\geq 1}$ in the spectrum $\sigma(y)$ of y ,*

such that

1) *the least upper bound of $(p_n)_{n\geq 1}$ in M is 1_M ,*

2) $y-\sum_{k=1}^{\infty}\lambda_k\, p_k \in \mathcal{J}$ *and* $\Big\| y-\sum_{k=1}^{\infty}\lambda_k\, p_k\Big\| \leq \varepsilon$.

Proof. Repeating word for word the arguments from the first paragraph of the proof of Theorem 4, we get $a\in M$ with $0\leq a\leq 1_M$, a continuous function $f:[0,+\infty)\to\mathbb{C}$ and $\delta>0$, such that $y=f(a)$ and

$$(\diamond)\qquad 0\leq b\in M\,,\ \|a-b\|\leq\delta \implies \|f(a)-f(b)\|\leq\varepsilon\,.$$

Subtracting from a an appropriate positive multiple of 1_M and modifying f corrispondingly, if necessary, we can assume that $0\in\sigma(a)$.

Choose a sequence $\delta/3=\delta_1>\delta_2>\ \ldots\ >0$ which converges to 0. According to the upper semicontinuity of the spectrum, there exist

$$\begin{array}{ccccccc} & & \eta_1 & > & \eta_2 & > & \ldots & > & 0\\ & & \wedge & & \wedge & & & & \\ \delta/3 & = & \delta_1 & > & \delta_2 & > & \ldots & & \end{array}$$

such that the spectrum of every $b\in M$ with $\|a-b\|\leq\eta_k$ is contained in

$$U_{\delta_k}\big(\sigma(a)\big)=\big\{\mu\in\mathbb{C}\,;\ |\mu-\lambda(\mu)|<\delta_k \text{ for some } \lambda(\mu)\in\sigma(a)\big\}\,.$$

Arguing now again as in the proof of Theorem 4, we can construct a sequence $0=e_o\leq e_1\leq e_2\leq\ldots$ of projections in $\mathcal{J}$, whose least upper bound in M is 1_M, such that

$$\|e_k a - a e_k\|\leq 2^{-k-1}\eta_{k+1} \text{ for all } k\geq 1\,.$$

Setting then

$$\begin{aligned} b_o &= a\,,\\ b_n &= \sum_{k=1}^{n}(e_k-e_{k-1})a(e_k-e_{k-1})+(1_M-e_n)a(1_M-e_n)\,,\quad n\geq 1\,,\end{aligned}$$

we have

$$b_{n-1}-b_n=(1_M-e_{n-1})\cdot[e_n\,,\,e_n a-a e_n]\cdot(1_M-e_{n-1})\,,\quad n\geq 1\,,$$

so $\|b_{n-1}-b_n\|\leq 2^{-n}\eta_{n+1}\leq 2^{-n}\delta/3$ and $b_{n-1}-b_n\in\mathcal{J}$. Therefore the sequence $(b_n)_{n\geq 1}$ is norm convergent to some $b_\infty\in M^+$, for which $\|a-b_\infty\|\leq\delta/3$ and $a-b_\infty\in\mathcal{J}$.

We claim that

$$b_\infty=\sum_{k=1}^{\infty}(e_k-e_{k-1})a(e_k-e_{k-1})\,.$$

Indeed, since

$$b_n\,(e_k-e_{k-1})=(e_k-e_{k-1})\,b_n=(e_k-e_{k-1})a(e_k-e_{k-1})\,,\qquad n\geq k\geq 1\,,$$

by passing to the limit for $n\to\infty$ we get

$$b_\infty\,(e_k-e_{k-1})=(e_k-e_{k-1})\,b_\infty=(e_k-e_{k-1})a(e_k-e_{k-1})\,,\qquad k\geq 1\,.$$

Thus, taking into account that $\bigvee_{k=1}^{\infty}(e_k-e_{k-1})=\bigvee_{k=1}^{\infty}e_k=1_M$, the description (*) yields the desired equality.

We notice that, for every $k\geq 1$,

$$(\diamond\diamond)\qquad \sigma\big((e_k-e_{k-1})a(e_k-e_{k-1})\big)\subset U_{\delta_k}\big(\sigma(a)\big)\,.$$

Indeed, since the norm of

$$\begin{aligned} & a - \Big((e_k - e_{k-1}) a (e_k - e_{k-1}) + \big(1_{A^{**}} - (e_k - e_{k-1})\big) a \big(1_{A^{**}} - (e_k - e_{k-1})\big)\Big) \\ = & \Big[[e_k - e_{k-1}, a] \, , 1_{A^{**}} - (e_k - e_{k-1}) \Big] \end{aligned}$$

is majorized by $2\left(\|e_k a - a e_k\| + \|e_{k-1} a - a e_{k-1}\|\right) \leq 2\left(2^{-k-2}\eta_{k+1} + 2^{-k-1}\eta_k\right) < \eta_k$, by the choice of η_k we have

$$\begin{aligned} & \sigma\big((e_k - e_{k-1}) a (e_k - e_{k-1})\big) \\ \subset & \, \sigma\Big((e_k - e_{k-1}) a (e_k - e_{k-1}) + \big(1_{A^{**}} - (e_k - e_{k-1})\big) a \big(1_{A^{**}} - (e_k - e_{k-1})\big)\Big) \cup \{0\} \\ \subset & \, U_{\delta_k}\big(\sigma(a)\big) \, . \end{aligned}$$

For any $k \geq 1$, let $[r_1^{(k)}, r_2^{(k)}]$ denote the smallest compact interval in $\mathbb{R}$ containing the spectrum $\sigma\big((e_k - e_{k-1}) a (e_k - e_{k-1})\big)$. Choose

$$r_1^{(k)} = \mu_1^{(k)} < \, \ldots < \mu_j^{(k)} < \, \ldots < \mu_{j_k}^{(k)} = r_2^{(k)}$$

in $\sigma\big((e_k - e_{k-1}) a (e_k - e_{k-1})\big)$ such that $|\mu_j^{(k)} - \mu_{j-1}^{(k)}| \leq \eta_k$ for al $2 \leq j \leq j_k$. Then there exist mutually orthogonal projections $\big(p_j^{(k)}\big)_{1 \leq j \leq j_k}$ in $\mathcal{J}$ such that

$$\sum_{j=1}^{j_k} p_j^{(k)} = e_k - e_{k-1} \;\text{ and }\; \Big\| (e_k - e_{k-1}) a (e_k - e_{k-1}) - \sum_{j=1}^{j_k} \mu_j^{(k)} p_j^{(k)} \Big\| \leq \eta_k \; :$$

For example, we can set $p_j^{(k)} = e_j^{(k)} - e_{j+1}^{(k)}$, $1 \leq j \leq j_k$, where

$$e_j^{(k)} = \mathbf{s}\Big(\big((e_k - e_{k-1}) a (e_k - e_{k-1}) - \mu_j^{(k)} \, (e_k - e_{k-1})\big)_+\Big) \, , \qquad 1 \leq j \leq j_k$$

and $e_{j_k+1}^{(k)} = 0$ (see e.g. [S-Z], 9.9, Proposition 1). Using ($\diamond\diamond$), we can find for every $\mu_j^{(k)}$ some $\lambda_j^{(k)} \in \sigma(a)$ with $|\lambda_j^{(k)} - \mu_j^{(k)}| < \delta_k$ and then

$$\Big\| (e_k - e_{k-1}) a (e_k - e_{k-1}) - \sum_{j=1}^{j_k} \lambda_j^{(k)} p_j^{(k)} \Big\| \leq \eta_k + \delta_k < 2\,\delta_k \leq 2\,\delta/3 \, .$$

Now $\bigcup_{k=1}^{\infty} \big\{ p_j^{(k)} \, ; \, 1 \leq j \leq j_k \big\}$ consists of mutually orthogonal projections in M , whose least upper bound in M is 1_M , while $\bigcup_{k=1}^{\infty} \big\{ \lambda_j^{(k)} \, ; \, 1 \leq j \leq j_k \big\} \subset \sigma(a)$. Set $b = \displaystyle\sum_{k=1}^{\infty} \sum_{j=1}^{j_k} \lambda_j^{(k)} p_j^{(k)} \in M^+$. Then (**) yields

$$\begin{aligned} \|b_\infty - b\| &= \Big\| \sum_{k=1}^{\infty} (e_k - e_{k-1}) a (e_k - e_{k-1}) - \sum_{k=1}^{\infty} \sum_{j=1}^{j_k} \lambda_j^{(k)} p_j^{(k)} \Big\| \\ &= \sup_{k \geq 1} \Big\| (e_k - e_{k-1}) a (e_k - e_{k-1}) - \sum_{j=1}^{j_k} \lambda_j^{(k)} p_j^{(k)} \Big\| \leq 2\,\delta/3 \, , \end{aligned}$$

so $\|a - b\| \leq \|a - b_\infty\| + \|b_\infty - b\| \leq \delta/3 + 2\,\delta/3 = \delta$. On the other hand, since

$$\Big\| \underbrace{(e_k - e_{k-1}) a (e_k - e_{k-1}) - \sum_{j=1}^{j_k} \lambda_j^{(k)} p_j^{(k)}}_{\in \mathcal{J}} \Big\| < 2\,\delta_k \longrightarrow 0 \, ,$$

(***) implies that $b_\infty - b \in \mathcal{J}$, hence $a - b = (a - b_\infty) + (b_\infty - b) \in \mathcal{J}$.

Using the characterization (*), it is easy to deduce that

$$f(b) = \sum_{k=1}^{\infty} \sum_{j=1}^{j_k} f(\lambda_j^{(k)})\, p_j^{(k)} ,$$

where, by the Spectral Mapping Theorem, $\bigcup_{k=1}^{\infty} \{ f(\lambda_j^{(k)}) \, ; \, 1 \leq j \leq j_k \}$ is contained in $f\big(\sigma(a)\big) = \sigma\big(f(a)\big) = \sigma(y)$. On the other hand, $(\diamond)$ yields the norm estimation $\| y - f(b) \| = \| f(a) - f(b) \| \leq \varepsilon$. Finally, using $a - b \in \mathcal{J}$ and the Weierstrass Approximation Theorem, we infer also that $y - f(b) = f(a) - f(b) \in \mathcal{J}$.

□

If in the above theorem we are not requiring the norm estimation in 2), then the coefficients λ_k can be chosen even in the essential spectrum of y modulo $\mathcal{J}$:

Theorem 6. *Let M be a unital Rickart C^*-algebra, and $\mathcal{J}$ a norm-closed two-sided ideal of M, which contains a sequence of positive elements such that 1_M is the only projection in M majorizing the sequence. For any normal $y \in M$ there are*

- *a sequence $(p_k)_{k \geq 1}$ of mutually orthogonal projections in $\mathcal{J}$,*
- *a sequence $(\lambda_k)_{k \geq 1}$ in the spectrum $\sigma_{\mathcal{J}}(y)$ of the canonical image of y in the quotient C^*-algebra $M/\mathcal{J}$*

such that

1) *the least upper bound of $(p_n)_{n \geq 1}$ in M is 1_M,*
2) $y - \sum_{k=1}^{\infty} \lambda_k \, p_k \in \mathcal{J}$.

For the proof we need the next lifting result, which is essentially [Z], Proposition 2.1 :

Lemma 7. *Let M be a unital Rickart C^*-algebra, and $\mathcal{J}$ a norm-closed two-sided ideal of M. For any self-adjoint $a \in M$ there exists a self-adjoint $b \in M$ such that $\sigma(b) = \sigma_{\mathcal{J}}(b)$ and $a - b \in \mathcal{J}$.*

Proof. A moment's reflection shows that the proof of [Z], Proposition 2.1 works for M unital Rickart C^*-algebra instead of W^*-algebra.

□

Proof of Theorem 6. Repeating again the arguments from the first paragraph of the proof of Theorem 4, we get some $a \in M$ with $0 \leq a \leq 1_M$ and a continuous function $f : [0, +\infty) \to \mathbb{C}$ such that $y = f(a)$. Now, according to Lemma 7, there exists a self-adjoint $b \in M$ such that $\sigma(b) = \sigma_{\mathcal{J}}(b)$ and $a - b \in \mathcal{J}$. In particular, $\sigma(b) = \sigma_{\mathcal{J}}(a) \subset [0,1]$, and so $0 \leq b \leq 1_M$.

Let x denote the normal element $f(b)$. Using the Weierstrass Approximation Theorem, we infer that $y - x \in \mathcal{J}$, hence, by the Spectral Mapping Theorem, we have $\sigma(x) = f\big(\sigma(b)\big) = f\big(\sigma_{\mathcal{J}}(a)\big) = \sigma_{\mathcal{J}}(y)$. Now Theorem 5 yields the existence of

- a sequence $(p_k)_{k \geq 1}$ of mutually orthogonal projections in $\mathcal{J}$,
- a sequence $(\lambda_k)_{k \geq 1}$ in $\sigma(x) = \sigma_{\mathcal{J}}(y)$,

such that the least upper bound of $(p_n)_{n \geq 1}$ in M is 1_M and $x - \sum_{k=1}^{\infty} \lambda_k \, p_k \in \mathcal{J}$. Then $y - \sum_{k=1}^{\infty} \lambda_k \, p_k = (y - x) + \big(x - \sum_{k=1}^{\infty} \lambda_k \, p_k\big) \in \mathcal{J}$.

□

Let us say that a C^*-algebra A is σ-subunital if there exists a sequence $(b_n)_{n\geq 1}$ in A^+, whose least upper bound in $M(A)_h$ is $1_{A^{**}}$. Clearly, if A is σ-unital then it is σ-subunital. For commutative A the two notions coincide. However, if M is a countably decomposable type II_∞-factor and A is the norm-closed linear span of all finite projections of M, then A is not σ-unital (see [Ak-Ped], Prop. 4.5), but it is easily seen that it is σ-subunital.

We remark that the sequence $(b_n)_{n\geq 1}$ in the definition of the σ-subunitalness can be considered a kind of "approximate unit with respect to the order structure". Indeed, according to [S-Z], 9.14, the remark after Proposition 3, if the least upper bound of $(b_n)_{n\geq 1}$ in $M(A)_h$ is $1_{A^{**}}$ and $x\in M(A)$, then the least upper bound of the sequence $\big(x^* b_n x\big)_{n\geq 1}$ in $M(A)_h$ is x^*x.

By Theorems 5 and 6 we have :

Corollary. *Let A be a σ-subunital C^*-algebra, whose multiplier algebra $M(A)$ is a Rickart C^*-algebra. For any normal $y\in M(A)$ and any $\varepsilon>0$ there exist*

- *a sequence $(p_k)_{k\geq 1}$ of mutually orthogonal projections in A,*
- *a sequence $(\lambda_k)_{k\geq 1}$ in the spectrum $\sigma(y)$ of y,*

such that

1) *the least upper bound of $(p_n)_{n\geq 1}$ in $M(A)_h$ is $1_{A^{**}}$,*
2) $y-\sum\limits_{k=1}^{\infty}\lambda_k\, p_k\in A$ *and* $\Big\|\, y-\sum\limits_{k=1}^{\infty}\lambda_k\, p_k\Big\|\leq\varepsilon$.

Moreover, if we don't require the second inequality in 2), then the sequence $(\lambda_k)_{k\geq 1}$ can be chosen even in the spectrum of the canonical image of y in the corona algebra $C(A)=M(A)/A$.

□

In particular, the above corollary can be applied to $A=K(H)$, where H is a separable complex Hilbert space, in which case the series $\sum_{k=1}^{\infty}\lambda_k\, p_k$ converges even with respect to the strict topology of $M(A)=B(H)$. This is the statement of the classical Weyl-von Neumann-Berg-Sikonia Theorem, but convergence with respect to the strict topology is used also in its subsequent extensions to σ-unital C^*-algebras with real rank zero multiplier algebra (see e.g. [M], [Br-Ped], [Zh], [H-Ro], [L1], [L2], [L3]).

On the other hand, in the early extension from [Z] of the Weyl-von Neumann-Berg-Sikonia Theorem to the norm-closed linear span A of all finite projections of an arbitrary semifinite W^*-factor M, which for M of type II_∞ turns out to be not σ-unital, the series $\sum_{k=1}^{\infty}\lambda_k\, p_k$ is proved to converge only with respect to the s^*-topology. The reason, why here a weaker topology than the strict topology should be used, is given by Theorem 3: if M is a type II_∞ W^*-factor and we assume that a sum $\sum_{k=1}^{\infty}\lambda_k\, p_k$ with $p_k\in A$ is strictly convergent, then, according to Theorem 3, we must have $\sum_{k=1}^{\infty}\lambda_k\, p_k\in A$.

Appendix

We give here, for the convenience of the reader, a treatment of a set-theoretical result of T. Iwamura (see [Ma], Appendix II) and two applications to the theory of AW^*-algebras.

Proposition. *Let $I, \leq$ be an upward directed partially ordered uncountable set. Then, there exist a well order $\preccurlyeq$ on I and a family $(I_\iota)_{\iota \in I}$ of subsets of I such that*

- I_ι *is upward directed for every* $\iota \in I$,
- $\operatorname{card} I_\iota < \operatorname{card} I$, $\quad \iota \in I$,
- $I_{\iota_1} \subset I_{\iota_2}$ *whenever* $\iota_1 \prec \iota_2$,
- $\bigcup_{\iota \in I} I_\iota = I$.

Proof. By Zermelo's theorem there exists a well order $\preccurlyeq$ on I . We can choose it such that

$$(*) \qquad \operatorname{card} \{\iota' \in I \,;\, \iota' \prec \iota\} < \operatorname{card} I \text{ for every } \iota \in I .$$

Indeed, if there exists some $\iota \in I$ such that

$$\operatorname{card} \{\iota' \in I \,;\, \iota' \prec \iota\} = \operatorname{card} I ,$$

then there exists a smallest ι with respect to $\preccurlyeq$, having the above property. Choose for this ι a bijection

$$\Phi : I \longrightarrow \{\iota' \in I \,;\, \iota' \prec \iota\}$$

and replace $\preccurlyeq$ by the well order, according to which ι_1 less or equal to ι_2 means $\Phi(\iota_1) \preccurlyeq \Phi(\iota_2)$.

We notice that, I being infinite, (*) implies that I does not contain a largest element with respect to $\preccurlyeq$.

Let us denote

$$J_\iota = \{\iota' \in I ;\ \iota' \prec i\}, \quad \iota \in I .$$

Then

$$\operatorname{card} J_\iota < \operatorname{card} I , \quad \iota \in I ,$$
$$J_{\iota_1} \subset J_{\iota_2} \text{ whenever } \iota_1 \prec \iota_2$$
$$\bigcup_{\iota \in I} J_\iota = I .$$

On the other hand, $I, \leq$ being upward directed, we can choose for each finite $F \subset I$ some $\iota(F) \in I$ such that

$$\iota \leq \iota(F) \text{ for all } \iota \in F .$$

Denote for every $J \subset I$

$$D_1(J) = J \cup \{\iota(F) \,;\, F \subset J \text{ finite } \} .$$

We notice that

$$D_1(J) \text{ is finite for } J \text{ finite,}$$
$$\operatorname{card} D_1(J) = \operatorname{card} J \text{ for } J \text{ infinite}$$

and

$$D_1(J_1) \subset D_1(J_2) \text{ whenever } J_1 \subset J_2 .$$

Now we define by recursion

$$D_{n+1}(J) = D_1(D_n(J)) \supset D_n(J),\ n \geq 1 \text{ integer},$$
$$D_\omega(J) = \bigcup_{n\geq 1} D_n(J)\,.$$

Then

$$D_\omega(J) \text{ is countable for } J \text{ finite},$$
$$\text{card } D_\omega(J) = \text{card } J \text{ for } J \text{ infinite}$$

and

$$D_\omega(J_1) \subset D_\omega(J_2) \text{ whenever } J_1 \subset J_2\,.$$

Moreover, $D_\omega(J), \leq$ is upward directed for every $J \subset I$.

Now, putting

$$I_\iota = D_\omega(J_\iota), \quad \iota \in I\,,$$

it is easy to see that all conditions from the statement are satisfied.

□

The first corollary extends Lemma 3 (compare with [Be], §33, Exercise 1):

Corollary 1. *Let M be an AW^*-algebra, $f \in M$ a finite projection, and $(e_\iota)_{\iota\in I}$ an upward directed family of projections in M such that*

$$e_\iota \prec f \textit{ for all } \iota \in I\,.$$

Then

$$\bigvee_{\iota\in I} e_\iota \prec f\,.$$

Proof. The case of countable I can be easily reduced to Lemma 6. Indeed, choosing a cofinal sequence $\iota_1 \leq \iota_2 \leq \dots$ in I, we have

$$\bigvee_{\iota\in I} e_\iota = \bigvee_{n\geq 1} e_{\iota_n} = e_{\iota_1} \vee \bigvee_{n\geq 1} (e_{\iota_{n+1}} - e_{\iota_n})$$

and we can apply Lemma 3 to f and the family $e_{\iota_1}\,,\ e_{\iota_2} - e_{\iota_1}\,,\ e_{\iota_3} - e_{\iota_2}\,, \dots$.

For the proof in the general case let $f \in M$ be a finite projection and let us assume the existence of some upward directed family $(e_\iota)_{\iota\in I}$ of projections in M such that

$$e_\iota \prec f \text{ for all } \iota \in I\,,\ \text{ but } \bigvee_{\iota\in I} e_\iota \not\prec f\,.$$

Choose among all such families one with I of the smallest cardinality. By the first part of the proof I is then uncountable.

Let the well order $\preccurlyeq$ on I and the family $(I_\iota)_{\iota\in I}$ of subsets of I be as in the above proposition.

According to the minimality property of card I, we have

$$p_\iota = \bigvee_{\iota'\in I_\iota} e_{\iota'} \prec f, \quad \iota \in I\,.$$

On the other hand,

$$p_{\iota_1} \leq p_{\iota_2} \quad \text{whenever } \iota_1 \prec \iota_2 \,,$$
$$\bigvee_{\iota\in I} p_\iota = \bigvee_{\iota\in I} e_\iota \,.$$

Consequently, denoting

$$q_\iota = p_\iota - \bigvee_{\iota'\prec\iota} p_{\iota'} \leq p_\iota \,, \quad \iota \in I \,,$$

the projections $(q_\iota)_{\iota\in I}$ are mutually orthogonal and

$$\sum_{\iota\in F} q_\iota \prec f \text{ for any finite } F \subset I \,.$$

By Lemma 6 it follows that

$$\bigvee_{\iota\in I} q_\iota \prec f \,.$$

But

$$\bigvee_{\iota\in I} q_\iota = \bigvee_{\iota\in I} p_\iota = \bigvee_{\iota\in I} e_\iota \,.$$

Indeed, otherwise it would exist a smallest $\iota \in I$ with respect to $\preccurlyeq$ such that

$$(**) \qquad p_\iota \not\leq \bigvee_{\iota'\in I} q_{\iota'} \,.$$

But then we would have

$$\bigvee_{\iota''\prec\iota} p_{\iota''} \leq \bigvee_{\iota'\in I} q_{\iota'} \,,$$

which contradicts $(**)$.

□

For M an arbitrary AW^*-algebra and Z a commutative AW^*-algebra we call

$$\Phi : \{ e \in M \,;\, e \text{ projection } \} \to Z^+$$

normal if, for every upward directed family $(e_\iota)_\iota$ of projections in M, we have

$$\Phi\Big(\bigvee_\iota e_\iota\Big) = \sup \Phi(e_\iota) \,,$$

where sup denotes the least upper bound in Z^+. Clearly,

$$\Phi \text{ normal} \Rightarrow \Phi \text{ completely additive,}$$

but, using the above proposition similarly as in the proof of the Corollary 2, we get also the converse implication (which should be known, but for which we have no reference):

Corollary 2. *Let M, Z be AW^*-algebras, Z commutative, and $\Phi : \{ e \in M \,;\, e$ projection $\} \to Z^+$. Then*

$$\Phi \textit{ normal} \Leftrightarrow \Phi \textit{ completely additive.}$$

In particular, the centre valued dimension function of a finite AW^*-algebra is normal (see [Be], §33, Exercise 4). Also, if M is a discrete AW^*-algebra and $e \in M$ is an abelian projection of central support 1_M, then the map Φ_e considered in the proof of Theorem 1 (on the abelian strict closure in discrete AW^*-algebras) is normal on the projection lattice of M.

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